

Generalizing the optimized gradient method for smooth convex minimization

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Abstract The optimized gradient method (OGM) [11] was recently developed by optimizing the step coefficients of first-order methods with respect to the function value [5]. This OGM has a per-iteration computational cost that is similar to Nesterov’s fast gradient method (FGM) [16], and satisfies a worst-case convergence bound of the function value that is twice smaller than that of FGM. Moreover, OGM was shown recently to achieve the optimal cost function complexity bound of first-order methods (with either fixed or dynamic step sizes) for smooth convex minimization in [4]. Considering that OGM is superior to the widely used FGM for smooth convex minimization with respect to the worst-case performance of the function decrease, it is desirable to further understand the formulation and convergence analysis of OGM. Therefore, this paper studies a generalized formulation of OGM and its convergence analysis in terms of both the function value and the gradient. We then optimize the step coefficients of first-order methods in terms of the rate of decrease of the norm of the gradient. This analysis leads to a new algorithm called OGM-OG that has the best known analytical worst-case bound on the decrease of the gradient norm among fixed-step first-order methods.

1 Introduction

First-order methods are favorable when solving large-scale problem, because their computational complexity per iteration depends mildly on the problem dimension. Both the heavy-ball method [20] and Nesterov’s fast gradient method (FGM) [16] are widely used fast and computationally efficient first-order methods. In particular, FGM achieves the optimal rate $O(1/N^2)$ for decreasing smooth convex functions after N iterations and thus has been celebrated in many (large-scale) applications.

While FGM enjoys the optimal rate for smooth convex minimization, Drori and Teboulle [5] explored whether there exists a first-order method with a worst-case cost function convergence bound that is lower than that of FGM by a constant. They first casted a worst-case convergence analysis into an optimization problem called Performance Estimation Problem (PEP) [5] that computes the maximal absolute cost function inaccuracy over all possible inputs (functions) to the optimization algorithm (see [6, 11, 12, 13, 22, 23] for its extensions). Moreover, Drori and Teboulle [5] optimized numerically the step coefficients of first-order methods using PEP for smooth convex minimization, and found an algorithm whose bound is lower than that of FGM, but it remained computationally and memory-wise expensive to use for large-scale problems. Building on their work, the authors [11, 12] found an equivalent computationally and memory-wise efficient version, named the optimized gradient method (OGM), satisfying an analytical bound that is twice smaller than that of FGM. Recently, Drori [4] showed that OGM has the optimal worst-case cost function convergence bound for large-dimensional smooth convex minimization, among a general class of first-order methods with either fixed or dynamic step sizes. This OGM has been

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numerically extended for nonsmooth composite convex problems in [22]. Interestingly this OGM-type acceleration was already studied in the context of a proximal point method [9, Appendix].

Considering that OGM has such optimality and is superior to FGM in the worst-case scenario for smooth convex minimization, it is desirable to understand its (generalized) formulation and convergence analysis for both the cost function and the gradient. Therefore, this paper uses the PEP approach to propose a generalized version of OGM (GOGM) and analyze its convergence rate. The results complement the convergence analysis of the OGM [11, 12], and expands our understanding of OGM-type first-order methods.

It is known that analyzing the convergence rate of the gradient norm is important in addition to analyzing the convergence of the cost function value when dealing with dual problems (see *e.g.*, [15, 19]). Nesterov [19] thus proposed a method that has a fast $O(1/N^{\frac{3}{2}})$ rate for the gradient norm decrease, which performs FGM for the first half of the iterations and uses GM for the remaining iterations. Nesterov considered such modified method (that requires selecting the total number of iterations N in advance, unlike FGM), because FGM has not yet been shown to satisfy a rate $O(1/N^{\frac{3}{2}})$ for decreasing the gradient norm; this paper proves that FGM in fact does have that rate. For further acceleration, we optimize the step coefficients of first-order methods with respect to the gradient norm convergence rate using PEP and propose an algorithm named OGM-OG (OG for optimized over a gradient) that belongs to the GOGM class and has the best known analytical worst-case bound on rate of decrease of the gradient norm among fixed-step first-order methods.

Section 2 defines the smooth convex problem and the first-order methods. Section 3 reviews and discusses convergence analyses of GM, FGM, and OGM for both the function value and the gradient norm. Section 3 also reviews first-order methods that guarantee an $O(1/N^{\frac{3}{2}})$ rate for the gradient decrease. Section 4 reviews Drori and Teboulle's cost function form of PEP [5] and reviews how the OGM [11] is derived using such PEP. Section 4 then proposes a generalized version of OGM (GOGM) using the cost function form of PEP, and Section 5 provides a worst-case gradient norm bound for the GOGM using the gradient form of PEP. Then, Section 5 optimizes the step coefficients using the gradient form of PEP and proposes the OGM-OG that belongs to the GOGM. Section 5 also proves that FGM decreases the gradient norm with a rate $O(1/N^{\frac{3}{2}})$. Section 6 and Section 7 provide discussion and conclusion.

2 Smooth convex problem and first-order methods

We focus on the following smooth convex minimization problem

$$\min_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x}), \quad (\text{M})$$

where the following additional conditions are assumed:

- $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is a convex function of the type $\mathcal{C}_L^{1,1}(\mathbb{R}^d)$, i.e., continuously differentiable with Lipschitz continuous gradient:

$$\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d, \quad (1)$$

where $L > 0$ is the Lipschitz constant.

- The optimal set $X_*(f) = \arg \min_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x})$ is nonempty, i.e., the problem (M) is solvable.

We use $\mathcal{F}_L(\mathbb{R}^d)$ to denote the class of functions that satisfy the above conditions. We also assume that the distance between an initial point \mathbf{x}_0 and an optimal solution $\mathbf{x}_* \in X(f)$ is bounded by some $R > 0$, i.e., $\|\mathbf{x}_0 - \mathbf{x}_*\| \leq R$.

To solve (M), we consider the following class of *fixed-step* first-order methods (FSFOM), where the update step at $(i + 1)$ th iteration is a weighted sum of the previous and current gradients $\{\nabla f(\mathbf{x}_k)\}_{k=0}^i$ with the fixed constant step coefficients $\{h_{i+1,k}\}_{k=0}^i$. This class FSFOM includes GM, the heavy-ball method, FGM, OGM, and the methods proposed in this paper.

Algorithm Class FSFOMInput: $f \in \mathcal{F}_L(\mathbb{R}^d)$, $\mathbf{x}_0 \in \mathbb{R}^d$.For $i = 0, \dots, N-1$

$$\mathbf{x}_{i+1} = \mathbf{x}_i - \frac{1}{L} \sum_{k=0}^i h_{i+1,k} \nabla f(\mathbf{x}_k).$$

3 Review of convergence analysis of FSFOM**3.1 Function value convergence analysis of FSFOM**

The simplest example of FSFOM is the following GM that only uses the current gradient and the Lipschitz constant L for the update.

Algorithm GMInput: $f \in \mathcal{F}_L(\mathbb{R}^d)$, $\mathbf{x}_0 \in \mathbb{R}^d$.For $i = 0, 1, \dots$

$$\mathbf{x}_{i+1} = \mathbf{x}_i - \frac{1}{L} \nabla f(\mathbf{x}_i)$$

This GM monotonically decreases the cost function [17] and satisfies the following tight¹ bound [5, Thm. 1], for any $i \geq 0$,

$$f(\mathbf{x}_i) - f(\mathbf{x}_*) \leq \frac{LR^2}{4i+2}. \quad (2)$$

Among the class FSFOM, the following two equivalent forms of FGM [16, 18] have been used widely because they decrease the cost function with the optimal rate $O(1/N^2)$.

Algorithm FGM1Input: $f \in \mathcal{F}_L(\mathbb{R}^d)$, $\mathbf{x}_0 = \mathbf{y}_0 \in \mathbb{R}^d$, $t_0 = 1$.For $i = 0, 1, \dots$

$$\mathbf{y}_{i+1} = \mathbf{x}_i - \frac{1}{L} \nabla f(\mathbf{x}_i)$$

$$t_{i+1} = \frac{1 + \sqrt{1 + 4t_i^2}}{2},$$

$$\mathbf{x}_{i+1} = \mathbf{y}_{i+1} + \frac{t_i - 1}{t_{i+1}} (\mathbf{y}_{i+1} - \mathbf{y}_i)$$

Algorithm FGM2Input: $f \in \mathcal{F}_L(\mathbb{R}^d)$, $\mathbf{x}_0 = \mathbf{y}_0 \in \mathbb{R}^d$, $t_0 = 1$.For $i = 0, 1, \dots$

$$\mathbf{y}_{i+1} = \mathbf{x}_i - \frac{1}{L} \nabla f(\mathbf{x}_i)$$

$$\mathbf{z}_{i+1} = \mathbf{x}_0 - \frac{1}{L} \sum_{k=0}^i t_k \nabla f(\mathbf{x}_k)$$

$$t_{i+1} = \frac{1 + \sqrt{1 + 4t_i^2}}{2},$$

$$\mathbf{x}_{i+1} = \left(1 - \frac{1}{t_{i+1}}\right) \mathbf{y}_{i+1} + \frac{1}{t_{i+1}} \mathbf{z}_{i+1}$$

Specifically, the FGM1 and FGM2 iterates satisfy the following bounds [11, 16, 18] for any $i \geq 1$:

$$f(\mathbf{y}_i) - f(\mathbf{x}_*) \leq \frac{LR^2}{2t_{i-1}^2} \leq \frac{2LR^2}{(i+1)^2}, \quad \text{and} \quad f(\mathbf{x}_i) - f(\mathbf{x}_*) \leq \frac{LR^2}{2t_i^2} \leq \frac{2LR^2}{(i+2)^2}, \quad (3)$$

¹ A ‘tight’ bound denotes an inequality bound which the equality exactly holds. For example, [5, Thm. 2] shows the tightness (exactness) of the bound (2).

where the parameter t_i satisfies

$$t_i^2 = \sum_{l=0}^i t_l \quad \text{and} \quad t_i \geq \frac{i+2}{2} \quad \text{for all } i. \quad (4)$$

A generalized form of FGM in [2] uses parameters t_i satisfying $t_0 = 1$ and $t_i^2 \leq t_{i-1}^2 + t_i$, including the choice $t_i = \frac{i+a}{a}$ for any $a \geq 2$. There is another generalized form of FGM in [18], and these generalized forms of FGM have been widely used and studied (*e.g.*, [1, 2, 21]). We similarly study a generalized form of the recently proposed OGM in this paper.

Inspired by [5] that optimized numerically the step coefficients over the cost function form of PEP, the following two equivalent forms of OGM were developed in [11], as reviewed in Section 4.

<p>Algorithm OGM1</p> <p>Input: $f \in \mathcal{F}_L(\mathbb{R}^d)$, $\mathbf{x}_0 = \mathbf{y}_0 \in \mathbb{R}^d$, $\theta_0 = 1$.</p> <p>For $i = 0, \dots, N-1$</p> $\mathbf{y}_{i+1} = \mathbf{x}_i - \frac{1}{L} \nabla f(\mathbf{x}_i)$ $\theta_{i+1} = \begin{cases} \frac{1+\sqrt{1+4\theta_i^2}}{2}, & i \leq N-2 \\ \frac{1+\sqrt{1+8\theta_i^2}}{2}, & i = N-1 \end{cases}$ $\mathbf{x}_{i+1} = \mathbf{y}_{i+1} + \frac{\theta_i - 1}{\theta_{i+1}} (\mathbf{y}_{i+1} - \mathbf{y}_i) \\ + \frac{\theta_i}{\theta_{i+1}} (\mathbf{y}_{i+1} - \mathbf{x}_i)$	<p>Algorithm OGM2</p> <p>Input: $f \in \mathcal{F}_L(\mathbb{R}^d)$, $\mathbf{x}_0 = \mathbf{y}_0 \in \mathbb{R}^d$, $\theta_0 = 1$.</p> <p>For $i = 0, \dots, N-1$</p> $\mathbf{y}_{i+1} = \mathbf{x}_i - \frac{1}{L} \nabla f(\mathbf{x}_i)$ $\mathbf{z}_{i+1} = \mathbf{x}_0 - \frac{1}{L} \sum_{k=0}^i 2\theta_k \nabla f(\mathbf{x}_k)$ $\theta_{i+1} = \begin{cases} \frac{1+\sqrt{1+4\theta_i^2}}{2}, & i \leq N-2 \\ \frac{1+\sqrt{1+8\theta_i^2}}{2}, & i = N-1 \end{cases}$ $\mathbf{x}_{i+1} = \left(1 - \frac{1}{\theta_{i+1}}\right) \mathbf{y}_{i+1} + \frac{1}{\theta_{i+1}} \mathbf{z}_{i+1}$
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The OGM iterates satisfy the following worst-case bounds [11, 12] for any $1 \leq i \leq N$:

$$f(\mathbf{y}_i) - f(\mathbf{x}_*) \leq \frac{LR^2}{4\theta_{i-1}^2} \leq \frac{LR^2}{(i+1)^2}, \quad (5)$$

and

$$f(\mathbf{x}_N) - f(\mathbf{x}_*) \leq \frac{LR^2}{2\theta_N^2} \leq \frac{LR^2}{(N+1)(N+1+\sqrt{2})}. \quad (6)$$

The parameter θ_i satisfies

$$\theta_i^2 = \begin{cases} \sum_{l=0}^{i+1} \theta_l, & i \leq N-1, \\ 2 \sum_{l=0}^{N-1} \theta_l + \theta_N, & i = N, \end{cases} \quad \text{and} \quad \theta_i \geq \begin{cases} \frac{i+2}{2}, & i \leq N-1, \\ \frac{i+1}{\sqrt{2}}, & i = N, \end{cases} \quad (7)$$

which is equivalent to t_i (4) except the final iteration. The bounds (5) and (6) of OGM are about twice smaller than the bounds (3) of FGM. Interestingly, the bound (6) on the final iterate \mathbf{x}_N is tight and satisfies the optimal worst-case convergence bound of general first-order methods including both FSFOM and dynamic-step first-order methods, when the condition $d \geq N+1$ holds [4].

Both the additional term $\frac{\theta_i}{\theta_{i+1}} (\mathbf{y}_{i+1} - \mathbf{x}_i)$ of OGM1 and the additional constant 2 for the update of \mathbf{z}_i of OGM2, compared to FGM1 and FGM2 respectively, with the parameter θ_N make OGM optimal (for $d \geq N+1$) and thus OGM decreases the cost function faster than FGM in the worst-case. Such formulation and the resulting fast convergence speed are interesting, and thus our main goal of this paper is to generalize the form of OGM and analyze the convergence rate of such generalized OGM in terms of both the function value and the gradient norm, complementing the bounds (5) and (6) on the function value of OGM.

We are often interested in the convergence behavior of the gradient norm, and next section analyzes the convergence of the gradient of FSFOM.

3.2 Gradient norm convergence analysis of FSFOM

When tackling dual problems, it is known that the gradient norm convergence rate is important in addition to the function value convergence rate (see *e.g.*, [15, 19]). One simple way to find a (loose) bound for the gradient norm is to use the well-known convex inequality for convex functions with L -Lipschitz continuous gradients [17]:

$$\frac{1}{2L} \|\nabla f(\mathbf{x})\|^2 \leq f(\mathbf{x}) - f\left(\mathbf{x} - \frac{1}{L} \nabla f(\mathbf{x})\right) \leq f(\mathbf{x}) - f(\mathbf{x}_*), \quad \forall \mathbf{x} \in \mathbb{R}^d, \quad (8)$$

as discussed in [17, 23]. Combining the bounds (2), (3) and the inequality (8), for any $i \geq 1$, the GM iterates satisfy

$$\|\nabla f(\mathbf{x}_i)\| \leq \sqrt{2L(f(\mathbf{x}_i) - f(\mathbf{x}_*))} \leq \frac{LR}{\sqrt{2i+1}}, \quad (9)$$

and the iterates of FGM satisfy

$$\|\nabla f(\mathbf{y}_i)\| \leq \frac{LR}{t_{i-1}} \leq \frac{2LR}{i+1}, \quad \text{and} \quad \|\nabla f(\mathbf{x}_i)\| \leq \frac{LR}{t_i} \leq \frac{2LR}{i+2}. \quad (10)$$

Similarly for any $1 \leq i \leq N$, the OGM iterates with the bounds (5) and (6) satisfy

$$\|\nabla f(\mathbf{y}_i)\| \leq \frac{LR}{\sqrt{2}\theta_{i-1}} \leq \frac{\sqrt{2}LR}{i+1}, \quad \text{and} \quad \|\nabla f(\mathbf{x}_N)\| \leq \frac{LR}{\theta_N} \leq \frac{\sqrt{2}LR}{N+1}. \quad (11)$$

Unfortunately, using the inequality (8) provides at best an $O(1/N)$ bound due to the optimal rate $O(1/N^2)$ of the function decrease.

Using a different method, Nesterov [19] derived a better $O(1/N)$ bound for the gradient norm of GM, as reviewed in next section. While the bounds (9), (10), and (11) are not guaranteed to be tight, the next section shows that the gradient bound (11) on the final iterate \mathbf{x}_N of OGM is in fact tight and thus has the same disappointingly slow $O(1/N)$ worst-case bound on the gradient norm as GM.

3.2.1 FSFOM with rate $O(1/N)$ for decreasing the gradient norm

This section uses the following lemma stating that GM monotonically decreases the gradient.

Lemma 1 [15, Lemma 2.4] *The GM monotonically decreases the gradient norm, i.e.,*

$$\left\| \nabla f\left(\mathbf{x} - \frac{1}{L} \nabla f(\mathbf{x})\right) \right\| \leq \|\nabla f(\mathbf{x})\|. \quad (12)$$

The following theorem reviews a simple proof in [19] that provides a gradient norm bound for GM with rate $O(1/N)$ that is smaller than (9), where [15, Thm. 6.1] additionally considers Lemma 1.

Theorem 1 [15, Thm. 6.1], [19] *Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be $\mathcal{F}_L(\mathbb{R}^d)$ and let $\mathbf{x}_0, \dots, \mathbf{x}_N \in \mathbb{R}^d$ be generated by Algorithm GM. Then for any $N \geq 1$,*

$$\min_{i \in \{0, \dots, N\}} \|\nabla f(\mathbf{x}_i)\| = \|\nabla f(\mathbf{x}_N)\| \leq \frac{\sqrt{2}LR}{\sqrt{N(N+2)}}. \quad (13)$$

Proof Lemma 1 implies the first equality in (13). Let $m = \lfloor \frac{N}{2} \rfloor$, and we have

$$\begin{aligned} \frac{LR^2}{4m+2} &\stackrel{(2)}{\geq} f(\mathbf{x}_m) - f(\mathbf{x}_*) \stackrel{(8)}{\geq} f(\mathbf{x}_{N+1}) - f(\mathbf{x}_*) + \frac{1}{2L} \sum_{i=m}^N \|\nabla f(\mathbf{x}_i)\|^2 \\ &\stackrel{(12)}{\geq} \frac{N-m+1}{2L} \|\nabla f(\mathbf{x}_N)\|^2, \end{aligned}$$

which is equivalent to (13) using $m \geq \frac{N-1}{2}$ and $N-m \geq \frac{N}{2}$.

Inspired by the conjecture in [23, Section 4.1.3], the following theorem shows that the $O(1/N)$ rate of the gradient norm bound (13) of GM is tight up to a constant.

Theorem 2 Let $\mathbf{x}_0, \dots, \mathbf{x}_N \in \mathbb{R}^d$ be generated by Algorithm GM. Then for any $N \geq 1$,

$$\frac{LR}{N+1} \leq \max_{f \in \mathcal{F}_L(\mathbb{R}^d)} \min_{i \in \{0, \dots, N\}} \|\nabla f(\mathbf{x}_i)\| \leq \max_{f \in \mathcal{F}_L(\mathbb{R}^d)} \|\nabla f(\mathbf{x}_N)\|, \quad (14)$$

where the first lower inequality in (14) is achieved by the following function in $\mathcal{F}_L(\mathbb{R}^d)$:

$$\psi(\mathbf{x}) = \begin{cases} \frac{LR}{N+1} \|\mathbf{x}\| - \frac{LR}{2(N+1)^2}, & \|\mathbf{x}\| \geq \frac{R}{N+1}, \\ \frac{L}{2} \|\mathbf{x}\|^2, & \|\mathbf{x}\| < \frac{R}{N+1}. \end{cases} \quad (15)$$

Proof Starting from $\mathbf{x}_0 = R\boldsymbol{\nu}$, where $\boldsymbol{\nu}$ is a unit vector, the GM iterates are

$$\mathbf{x}_i = \left(1 - \frac{i}{N+1}\right) R\boldsymbol{\nu}, \quad \nabla \psi(\mathbf{x}_i) = \frac{LR}{N+1}, \quad i = 0, \dots, N,$$

which implies the inequality (14).

We next show that the bound (11) for the final iterate \mathbf{x}_N of OGM is tight and its worst-case function is a simple quadratic function. Note that OGM was derived by optimizing bounds on the cost function decrease and its behavior in terms of gradient norms was not investigated previously.

Theorem 3 Let $\mathbf{x}_0, \dots, \mathbf{x}_N \in \mathbb{R}^d$ be generated by Algorithm OGM. Then for any $N \geq 1$,

$$\max_{f \in \mathcal{F}_L(\mathbb{R}^d)} \min_{i \in \{0, \dots, N\}} \|\nabla f(\mathbf{x}_i)\| = \max_{f \in \mathcal{F}_L(\mathbb{R}^d)} \|\nabla f(\mathbf{x}_N)\| = \frac{LR}{\theta_N} \leq \frac{\sqrt{2}LR}{N+1}, \quad (16)$$

where the worst-case function for OGM in terms of the gradient norm is the quadratic function $\phi(\mathbf{x}) = \frac{L}{2} \|\mathbf{x}\|^2$ in $\mathcal{F}_L(\mathbb{R}^d)$.

Proof See Appendix.

Comparing (13) and (16), we see that GM and OGM have essentially similar worst-case gradient norm bounds. This is an interesting dilemma where OGM is the fastest FSFOM in terms of the worst-case cost function bound, but is as slow as GM in terms of the gradient norm. Therefore the specific goal of this paper is to study optimizing the step coefficients of FSFOM using PEP with respect to the gradient norm in Section 5.4.

We next discuss the FSFOM in [19] that decreases the gradient norm with a faster $O(1/N^{\frac{3}{2}})$ rate.

3.2.2 FSFOM with rate $O(1/N^{\frac{3}{2}})$ for decreasing the gradient norm

Searching for FSFOM that minimizes the gradient norm faster than the $O(1/N)$ rate of GM (and OGM), Nesterov [19] proposed to perform FGM for the first $\lfloor \frac{N}{2} \rfloor$ iterations, and GM for the remaining iterations. He showed that this method satisfies a fast rate $O(1/N^{\frac{3}{2}})$ for decreasing the gradient norm, which we denote FGM-H (H for half). In [15, 19], FGM-H was used to solve dual problems where fast convergence rate of the gradient is particularly important. To pursue faster convergence (in terms of the constant factor), we consider another variant that performs OGM for the first $\lfloor \frac{N}{2} \rfloor$ iterations and GM for the remaining iterations, which we denote OGM-H.

Algorithm OGM-H

Input: $f \in \mathcal{F}_L(\mathbb{R}^d)$, $\mathbf{x}_0 = \mathbf{y}_0 \in \mathbb{R}^d$, $\vartheta_0 = 1$.

For $i = 0, \dots, N-1$

$$\mathbf{y}_{i+1} = \mathbf{x}_i - \frac{1}{L} \nabla f(\mathbf{x}_i)$$

$$\vartheta_{i+1} = \begin{cases} \frac{1 + \sqrt{1 + 4\vartheta_i^2}}{2}, & i \leq \lfloor \frac{N}{2} \rfloor - 2 \\ \frac{1 + \sqrt{1 + 8\vartheta_i^2}}{2}, & i = \lfloor \frac{N}{2} \rfloor - 1 \end{cases}$$

$$\mathbf{x}_{i+1} = \begin{cases} \mathbf{y}_{i+1} + \frac{\vartheta_i - 1}{\vartheta_{i+1}} (\mathbf{y}_{i+1} - \mathbf{y}_i) + \frac{\vartheta_i}{\vartheta_{i+1}} (\mathbf{y}_{i+1} - \mathbf{x}_i), & i \leq \lfloor \frac{N}{2} \rfloor - 1, \\ \mathbf{y}_{i+1}, & \text{otherwise.} \end{cases}$$

The following theorem bounds the gradient norm of the OGM-H iterates, based on the proof in [15, 19] for the gradient norm bound of the FGM-H iterates. The FGM-H bound in [15, 19] is asymptotically $\sqrt{2}$ -times larger than the bound (17) of OGM-H.

Theorem 4 Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be $\mathcal{F}_L(\mathbb{R}^d)$ and let $\mathbf{x}_0, \dots, \mathbf{x}_N \in \mathbb{R}^d$ be generated by Algorithm OGM-H. Then for any $N \geq 1$,

$$\min_{i \in \{0, \dots, N\}} \|\nabla f(\mathbf{x}_i)\| \leq \|\nabla f(\mathbf{x}_N)\| \leq \frac{4LR}{(N+1)\sqrt{N+2}}. \quad (17)$$

Proof Let $m = \lfloor \frac{N}{2} \rfloor$, and we have

$$\begin{aligned} \frac{LR^2}{2\vartheta_m^2} &\stackrel{(6)}{\geq} f(\mathbf{x}_m) - f(\mathbf{x}_*) \stackrel{(8)}{\geq} f(\mathbf{x}_{N+1}) - f(\mathbf{x}_*) + \frac{1}{2L} \sum_{i=m}^N \|\nabla f(\mathbf{x}_i)\|^2 \\ &\stackrel{(12)}{\geq} \frac{N-m+1}{2L} \|\nabla f(\mathbf{x}_N)\|^2, \end{aligned}$$

which is equivalent to (17) using $m \geq \frac{N-1}{2}$, $N-m \geq \frac{N}{2}$ and $\vartheta_m \geq \frac{m+1}{\sqrt{2}} \geq \frac{N+1}{2\sqrt{2}}$ that is implied by (7).

Both FGM-H and OGM-H require that one select the total number of iterations N in advance, which can be undesirable in practice. Nesterov [19] used such methods since, prior to this paper, it was unknown whether FGM (that does not require selecting N in advance) decreases the gradient norm with the rate $O(1/N^{\frac{3}{2}})$; this rate for the gradient norm of FGM is numerically conjectured by Taylor *et al.* [23]. The authors in [7, 14] derived an $O(1/N^{\frac{3}{2}})$ bound for variants of FGM with a step size that is strictly smaller than $\frac{1}{L}$ in FGM. Section 5.2 below uses the PEP to show for the first time the rate $O(1/N^{\frac{3}{2}})$ for the gradient decrease of the FGM. The bound (17) of OGM-H for decreasing the gradient is smaller than those in [7, 14], and Section 5.4 below shows that our proposed methods have bounds even lower than (17).

We next review how [5] and [11] optimized the step coefficients of the FSFOM class over the cost function form of PEP, leading to OGM. Then, we generalize the formulation and convergence analysis of OGM.

4 Relaxation and optimization of the cost function form of PEP

4.1 Review: Relaxation for the cost function form of PEP

The worst-case bound of the cost function for FSFOM with given step coefficients $\mathbf{h} := \{h_{i+1,k}\}$ corresponds to a solution of the following PEP problem [5, Problem (P)]:

$$\begin{aligned} \mathcal{B}_P(\mathbf{h}, N, d, L, R) &:= \max_{\substack{f \in \mathcal{F}_L(\mathbb{R}^d), \\ \mathbf{x}_0, \dots, \mathbf{x}_N \in \mathbb{R}^d, \\ \mathbf{x}_* \in X_*(f)}} f(\mathbf{x}_N) - f(\mathbf{x}_*) \quad (P) \\ \text{s.t. } \mathbf{x}_{i+1} &= \mathbf{x}_i - \frac{1}{L} \sum_{k=0}^i h_{i+1,k} \nabla f(\mathbf{x}_k), \quad i = 0, \dots, N-1, \\ \|\mathbf{x}_0 - \mathbf{x}_*\| &\leq R. \end{aligned}$$

Since problem (P) is impractical to solve due to its functional constraint, it is relaxed as follows in [5, Problem (Q)]:

$$\begin{aligned} \mathcal{B}_{P1}(\mathbf{h}, N, d, L, R) &:= \max_{\substack{\mathbf{G} \in \mathbb{R}^{(N+1) \times d}, \\ \boldsymbol{\delta} \in \mathbb{R}^{N+1}}} LR^2 \delta_N \quad (P1) \\ \text{s.t. } \text{Tr}\{\mathbf{G}^\top \mathbf{A}_{i,j}(\mathbf{h}) \mathbf{G}\} &\leq \delta_i - \delta_j, \quad i < j = 0, \dots, N, \\ \text{Tr}\{\mathbf{G}^\top \mathbf{B}_{i,j}(\mathbf{h}) \mathbf{G}\} &\leq \delta_i - \delta_j, \quad j < i = 0, \dots, N, \\ \text{Tr}\{\mathbf{G}^\top \mathbf{C}_i \mathbf{G}\} &\leq \delta_i, \quad i = 0, \dots, N, \\ \text{Tr}\{\mathbf{G}^\top \mathbf{D}_i(\mathbf{h}) \mathbf{G} + \nu \mathbf{u}_i^\top \mathbf{G}\} &\leq -\delta_i, \quad i = 0, \dots, N, \end{aligned}$$

for any given unit vector $\boldsymbol{\nu} \in \mathbb{R}^d$, where $\mathbf{u}_i = \mathbf{e}_{i+1} \in \mathbb{R}^{N+1}$ is the $(i+1)$ th standard basis vector, and we denote $\mathbf{G} = [\mathbf{g}_0, \dots, \mathbf{g}_N]^\top \in \mathbb{R}^{(N+1) \times d}$ for $\mathbf{g}_i := \frac{1}{L\|\mathbf{x}_0 - \mathbf{x}_*\|} \nabla f(\mathbf{x}_i)$, and $\boldsymbol{\delta} = [\delta_0, \dots, \delta_N]^\top \in \mathbb{R}^{N+1}$ for $\delta_i := \frac{1}{L\|\mathbf{x}_0 - \mathbf{x}_*\|^2} (f(\mathbf{x}_i) - f(\mathbf{x}_*))$. Note that $\|\mathbf{g}_*\| = \delta_* = 0$ and $\text{Tr}\{\mathbf{G}^\top \mathbf{u}_i \mathbf{u}_j^\top \mathbf{G}\} = \langle \mathbf{g}_i, \mathbf{g}_j \rangle$ by definition. The matrices $\mathbf{A}_{i,j}(\mathbf{h}), \mathbf{B}_{i,j}(\mathbf{h}), \mathbf{C}_i, \mathbf{D}_i(\mathbf{h})$ are defined as

$$\begin{cases} \mathbf{A}_{i,j}(\mathbf{h}) := \frac{1}{2}(\mathbf{u}_i - \mathbf{u}_j)(\mathbf{u}_i - \mathbf{u}_j)^\top + \frac{1}{2} \sum_{l=i+1}^j \sum_{k=0}^{l-1} h_{l,k}(\mathbf{u}_j \mathbf{u}_k^\top + \mathbf{u}_k \mathbf{u}_j^\top), \\ \mathbf{B}_{i,j}(\mathbf{h}) := \frac{1}{2}(\mathbf{u}_i - \mathbf{u}_j)(\mathbf{u}_i - \mathbf{u}_j)^\top - \frac{1}{2} \sum_{l=j+1}^i \sum_{k=0}^{l-1} h_{l,k}(\mathbf{u}_j \mathbf{u}_k^\top + \mathbf{u}_k \mathbf{u}_j^\top), \\ \mathbf{C}_i := \frac{1}{2} \mathbf{u}_i \mathbf{u}_i^\top, \\ \mathbf{D}_i(\mathbf{h}) := \frac{1}{2} \mathbf{u}_i \mathbf{u}_i^\top + \frac{1}{2} \sum_{j=1}^i \sum_{k=0}^{j-1} h_{j,k}(\mathbf{u}_i \mathbf{u}_k^\top + \mathbf{u}_k \mathbf{u}_i^\top). \end{cases} \quad (18)$$

In [5, Problem (Q')], problem (P1) is further relaxed by discarding some constraints as

$$\begin{aligned} \mathcal{B}_{P2}(\mathbf{h}, N, d, L, R) &:= \max_{\substack{\mathbf{G} \in \mathbb{R}^{(N+1) \times d}, \\ \boldsymbol{\delta} \in \mathbb{R}^{N+1}}} LR^2 \delta_N \\ \text{s.t. } &\text{Tr}\{\mathbf{G}^\top \mathbf{A}_{i-1,i}(\mathbf{h}) \mathbf{G}\} \leq \delta_{i-1} - \delta_i, \quad i = 1, \dots, N, \\ &\text{Tr}\{\mathbf{G}^\top \mathbf{D}_i(\mathbf{h}) \mathbf{G} + \boldsymbol{\nu} \mathbf{u}_i^\top \mathbf{G}\} \leq -\delta_i, \quad i = 0, \dots, N, \end{aligned} \quad (P2)$$

for any given unit vector $\boldsymbol{\nu} \in \mathbb{R}^d$. We explicitly illustrate this relaxation step from (P1) to (P2), because Section 5 uses a similar but different relaxation. Taylor *et al.* [23] avoided this step to analyze a tight bound of (P) (under a large-scale condition “ $d \geq N + 2$ ” [23, Thm. 5]); however, this relaxation step facilitates the analysis in [5, 11, 12] and in this paper.

Replacing $\max_{\mathbf{G}, \boldsymbol{\delta}} LR^2 \delta_N$ by $\min_{\mathbf{G}, \boldsymbol{\delta}} \{-\delta_N\}$ for convenience in (P2), the Lagrangian of the corresponding constrained minimization problem with dual variables $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)^\top \in \mathbb{R}^N$ and $\boldsymbol{\tau} = (\tau_0, \dots, \tau_N)^\top \in \mathbb{R}^{N+1}$ for the first and second constraint inequalities of (P2) respectively becomes

$$\mathcal{L}(\mathbf{G}, \boldsymbol{\delta}, \boldsymbol{\lambda}, \boldsymbol{\tau}; \mathbf{h}) = -\delta_N + \sum_{i=1}^N \lambda_i (\delta_i - \delta_{i-1}) + \sum_{i=0}^N \tau_i \delta_i + \text{Tr}\{\mathbf{G}^\top \mathbf{S}(\mathbf{h}, \boldsymbol{\lambda}, \boldsymbol{\tau}) \mathbf{G} + \boldsymbol{\nu} \boldsymbol{\tau}^\top \mathbf{G}\}, \quad (19)$$

where

$$\mathbf{S}(\mathbf{h}, \boldsymbol{\lambda}, \boldsymbol{\tau}) := \sum_{i=1}^N \lambda_i \mathbf{A}_{i-1,i}(\mathbf{h}) + \sum_{i=0}^N \tau_i \mathbf{D}_i(\mathbf{h}). \quad (20)$$

Then, we have the following dual problem of (P2) that one could use to compute a valid upper bound of (P) using a semidefinite program (SDP) for given \mathbf{h} [5, Problem (DQ')]:

$$\mathcal{B}_D(\mathbf{h}, N, L, R) := \min_{\substack{(\boldsymbol{\lambda}, \boldsymbol{\tau}) \in \Lambda, \\ \gamma \in \mathbb{R}}} \left\{ \frac{1}{2} LR^2 \gamma : \begin{pmatrix} \mathbf{S}(\mathbf{h}, \boldsymbol{\lambda}, \boldsymbol{\tau}) & \frac{1}{2} \boldsymbol{\tau} \\ \frac{1}{2} \boldsymbol{\tau}^\top & \frac{1}{2} \gamma \end{pmatrix} \succeq 0 \right\}, \quad (D)$$

where

$$\Lambda = \left\{ (\boldsymbol{\lambda}, \boldsymbol{\tau}) \in \mathbb{R}_+^{2N+1} : \begin{aligned} &\tau_0 = \lambda_1, \quad \lambda_N + \tau_N = 1, \\ &\lambda_i - \lambda_{i+1} + \tau_i = 0, \quad i = 1, \dots, N-1, \end{aligned} \right\}. \quad (21)$$

The next section reviews the analytical solution to this upper bound (D) for OGM (and later for the generalized OGM), instead of using a numerical SDP solver.

4.2 Review: Optimizing step coefficients over the cost function form of PEP

Drori and Teboulle [5] optimized numerically the step coefficients \mathbf{h} over the simple SDP problem (D) as follows [5, Problem (BIL)]:

$$\hat{\mathbf{h}}_D := \arg \min_{\mathbf{h} \in \mathbb{R}^{N(N+1)/2}} \mathcal{B}_D(\mathbf{h}, N, L, R). \quad (HD)$$

The problem (HD) is bilinear, and a convex relaxation technique [5, Thm. 3]² was further employed to make it solvable using numerical methods.

In [11, Lemma 4], we solved (HD) analytically yielding the optimized step coefficients

$$h_{i+1,k} = \begin{cases} \frac{1}{\theta_{i+1}} \left(2\theta_k - \sum_{j=k+1}^i h_{j,k} \right), & k = 0, \dots, i-1, \\ 1 + \frac{2\theta_i-1}{\theta_{i+1}}, & k = i. \end{cases} \quad (22)$$

for θ_i in (7). Fortuitously, the optimized coefficients (22) lead to equivalent computationally efficient OGM1 and OGM2 forms [11, Prop. 3, 4 and 5], and the bound (6) of OGM is implied by [11, Lemma 4]. Recently, Drori's complexity analysis in [4] showed that the OGM is optimal for $d \geq N+1$, implying that optimizing over the relaxed bound (D) in (HD) (for simplicity) is equivalent to optimizing over the exact cost function bound (P) when $d \geq N+1$.

While one could use a SDP solver to compute a numerical bound (D) for any FSFOM, deriving an analytical bound using (D) is found to be difficult for the primary sequence $\{\mathbf{y}_i\}$ of OGM. Therefore, we devised a new relaxed bound in [12] similar to (D), which we review next.

4.3 Review: Another cost function form of relaxed PEP for the primary sequence of OGM

An upper bound of the worst-case bound on $f(\mathbf{y}_{N+1}) - f(\mathbf{x}_*)$ for FSFOM with step coefficients \mathbf{h} and $\mathbf{y}_{N+1} = \mathbf{x}_N - \frac{1}{L}f(\mathbf{x}_N)$ could be computed using (D) by a SDP solver. However, we found it difficult to find its analytical bound for \mathbf{h} of OGM, and thus we derived the following alternate upper bound on $f(\mathbf{y}_{N+1}) - f(\mathbf{x}_*)$ in [12, Problem (D')]:

$$\mathcal{B}_{D'}(\mathbf{h}, N, L, R) := \min_{\substack{(\boldsymbol{\lambda}, \boldsymbol{\tau}) \in \mathcal{A}, \\ \boldsymbol{\gamma} \in \mathbb{R}}} \left\{ \frac{1}{2}LR^2\boldsymbol{\gamma} : \begin{pmatrix} S(\mathbf{h}, \boldsymbol{\lambda}, \boldsymbol{\tau}) + \frac{1}{2}\mathbf{u}_N\mathbf{u}_N^\top & \frac{1}{2}\boldsymbol{\tau} \\ \frac{1}{2}\boldsymbol{\tau}^\top & \frac{1}{2}\boldsymbol{\gamma} \end{pmatrix} \succeq 0 \right\}, \quad (D')$$

which led to the bound (5) for the primary sequence $\{\mathbf{y}_i\}$ of OGM in [12].

Similar to [11, Lemma 4], we found a feasible point of (D') in [12, Lemma 3.1], along with feasible step coefficients \mathbf{h} of FSFOM:

$$h_{i+1,k} = \begin{cases} \frac{1}{t_{i+1}} \left(2t_k - \sum_{j=k+1}^i h_{j,k} \right), & k = 0, \dots, i-1, \\ 1 + \frac{2t_i-1}{t_{i+1}}, & k = i. \end{cases} \quad (23)$$

for t_i in (4). Then, [12, Thm. 3.1] shows the bound (5) using [12, Lemma 3.1]. Note that the step coefficients (22) and (23) are identical except the final iteration, since t_i (4) and θ_i (7) are equivalent for $i < N$.

We are now done reviewing the portions of papers [11, 12] that are the ingredients for generalizing the OGM. We next specify feasible points of (D) and (D') that lead to a generalized version of OGM.

4.4 Feasible points of (D) and (D') for the generalized OGM

The following lemma specifies some feasible points of (D) that lead to our generalized OGM. This lemma reduces to [11, Lemma 4] (and the step coefficients (22) of OGM) when $\theta_i^2 = \Omega_i$ for all i .

Lemma 2 *For the following step coefficients:*

$$h_{i+1,k} = \begin{cases} \frac{\theta_{i+1}}{\Omega_{i+1}} \left(2\theta_k - \sum_{j=k+1}^i h_{j,k} \right), & i = 0, \dots, N-1, k = 0, \dots, i-1, \\ 1 + \frac{(2\theta_i-1)\theta_{i+1}}{\Omega_{i+1}}, & i = 0, \dots, N-1, k = i, \end{cases} \quad (24)$$

² [5, Thm. 3] has typos that are fixed in [11, Eq. (6.3)].

the choice of variables:

$$\lambda_i = \Omega_{i-1}\tau_0, \quad i = 1, \dots, N, \quad \tau_i = \begin{cases} \frac{2}{\Omega_N}, & i = 0, \\ \theta_i\tau_0 & i = 1, \dots, N-1, \\ \frac{\theta_N}{2}\tau_0, & i = N, \end{cases} \quad \gamma = \frac{1}{2}\tau_0 \quad (25)$$

is a feasible point of (D) for any choice of θ_i such that

$$\theta_0 = 1, \quad \theta_i > 0, \quad \text{and} \quad \theta_i^2 \leq \Omega_i := \begin{cases} \sum_{l=0}^i \theta_l, & i = 0, \dots, N-1, \\ 2 \sum_{l=0}^{N-1} \theta_l + \theta_N, & i = N. \end{cases} \quad (26)$$

Proof See Appendix.

The following lemma also specifies some feasible points of (D'), and this reduces to [12, Lemma 3.1] (and (23)) when $t_i^2 = T_i$ for all i .

Lemma 3 For the following step coefficients:

$$h_{i+1,k} = \begin{cases} \frac{t_{i+1}}{T_{i+1}} \left(2t_k - \sum_{j=k+1}^i h_{j,k} \right), & i = 0, \dots, N-1, \quad k = 0, \dots, i-1, \\ 1 + \frac{(2t_i-1)t_{i+1}}{T_{i+1}}, & i = 0, \dots, N-1, \quad k = i, \end{cases} \quad (27)$$

the choice of variables:

$$\lambda_i = T_{i-1}\tau_0, \quad i = 1, \dots, N, \quad \tau_i = \begin{cases} \frac{1}{T_N}, & i = 0, \\ t_i\tau_0 & i = 1, \dots, N, \end{cases} \quad \gamma = \frac{1}{2}\tau_0 \quad (28)$$

is a feasible point of (D') for any choice of t_i such that

$$t_0 = 1, \quad t_i > 0, \quad \text{and} \quad t_i^2 \leq T_i := \sum_{l=0}^i t_l. \quad (29)$$

Proof See Appendix.

Similar to the relationship between the step coefficients (22) and (23), the step coefficients (24) and (27) are identical (when $\theta_i = t_i$ for $i < N$) except for the final iteration, implying that the iterates $\{\mathbf{x}_i\}_{i=0}^{N-1}$ of FSFOM with both (24) and (27) are equivalent; only the final iterate \mathbf{x}_N is different. The step coefficients (22) and (23) lead to computationally efficient equivalent OGM forms, and similarly we next provide computationally efficient generalized forms of OGM that correspond to FSFOM with (24) and (27), and we analyze their cost function convergence bounds.

4.5 Generalized OGM

This section proposes a generalized OGM using lemmas 2 and 3. FSFOM with the step coefficients (24) has the following two equivalent efficient generalized forms of OGM, named GOGM1 and GOGM2, which reduce to the standard OGM when $\theta_i^2 = \Omega_i$ for all i .

<p>Algorithm GOGM1</p> <p>Input: $f \in \mathcal{F}_L(\mathbb{R}^d)$, $\mathbf{x}_0 = \mathbf{y}_0 \in \mathbb{R}^d$, $\theta_0 = \Omega_0 = 1$.</p> <p>For $i = 0, \dots, N-1$</p> $\mathbf{y}_{i+1} = \mathbf{x}_i - \frac{1}{L} \nabla f(\mathbf{x}_i)$ <p>Choose $\theta_{i+1} > 0$ s.t. $\theta_{i+1}^2 \leq \Omega_{i+1}$, where</p> $\Omega_{i+1} = \begin{cases} \sum_{l=0}^{i+1} \theta_l, & i \leq N-2, \\ 2 \sum_{l=0}^{N-1} \theta_l + \theta_N, & i = N-1. \end{cases}$ $\mathbf{x}_{i+1} = \mathbf{y}_{i+1} + \frac{(\Omega_i - \theta_i)\theta_{i+1}}{\theta_i \Omega_{i+1}} (\mathbf{y}_{i+1} - \mathbf{y}_i) \\ + \frac{(2\theta_i^2 - \Omega_i)\theta_{i+1}}{\theta_i \Omega_{i+1}} (\mathbf{y}_{i+1} - \mathbf{x}_i)$	<p>Algorithm GOGM2</p> <p>Input: $f \in \mathcal{F}_L(\mathbb{R}^d)$, $\mathbf{x}_0 = \mathbf{y}_0 \in \mathbb{R}^d$, $\theta_0 = \Omega_0 = 1$.</p> <p>For $i = 0, \dots, N-1$</p> $\mathbf{y}_{i+1} = \mathbf{x}_i - \frac{1}{L} \nabla f(\mathbf{x}_i)$ $\mathbf{z}_{i+1} = \mathbf{x}_0 - \frac{1}{L} \sum_{k=0}^i 2\theta_k \nabla f(\mathbf{x}_k)$ <p>Choose $\theta_{i+1} > 0$ s.t. $\theta_{i+1}^2 \leq \Omega_{i+1}$, where</p> $\Omega_{i+1} = \begin{cases} \sum_{l=0}^{i+1} \theta_l, & i \leq N-2, \\ 2 \sum_{l=0}^{N-1} \theta_l + \theta_N, & i = N-1. \end{cases}$ $\mathbf{x}_{i+1} = \left(1 - \frac{\theta_{i+1}}{\Omega_{i+1}}\right) \mathbf{y}_{i+1} + \frac{\theta_{i+1}}{\Omega_{i+1}} \mathbf{z}_{i+1}$
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Proposition 1 *The sequence $\{\mathbf{x}_0, \dots, \mathbf{x}_N\}$ generated by Algorithm FSFOM with (24) is identical to the corresponding sequence generated by Algorithm GOGM1 and GOGM2.*

Proof See Appendix. Note that this proof is independent of the choice of θ_i and Ω_i .

Because the proof of Prop. 1 for FSFOM with step coefficients (24) is independent of the choice of θ_i and Ω_i , it is straightforward to show that FSFOM with step coefficients (27) has the following two efficient equivalent forms, named GOGM1' and GOGM2', which reduce to the forms in [12, Algorithm OGM1' and OGM2'] when $t_i^2 = T_i$ for all i .

<p>Algorithm GOGM1'</p> <p>Input: $f \in \mathcal{F}_L(\mathbb{R}^d)$, $\mathbf{x}_0 = \mathbf{y}_0 \in \mathbb{R}^d$, $t_0 = T_0 = 1$.</p> <p>For $i = 0, \dots, N-1$</p> $\mathbf{y}_{i+1} = \mathbf{x}_i - \frac{1}{L} \nabla f(\mathbf{x}_i)$ <p>Choose $t_{i+1} > 0$ s.t. $t_{i+1}^2 \leq T_{i+1} = \sum_{l=0}^{i+1} t_l$</p> $\mathbf{x}_{i+1} = \mathbf{y}_{i+1} + \frac{(T_i - t_i)t_{i+1}}{t_i T_{i+1}} (\mathbf{y}_{i+1} - \mathbf{y}_i) \\ + \frac{(2t_i^2 - T_i)t_{i+1}}{t_i T_{i+1}} (\mathbf{y}_{i+1} - \mathbf{x}_i)$	<p>Algorithm GOGM2'</p> <p>Input: $f \in \mathcal{F}_L(\mathbb{R}^d)$, $\mathbf{x}_0 = \mathbf{y}_0 \in \mathbb{R}^d$, $t_0 = T_0 = 1$.</p> <p>For $i = 0, \dots, N-1$</p> $\mathbf{y}_{i+1} = \mathbf{x}_i - \frac{1}{L} \nabla f(\mathbf{x}_i)$ $\mathbf{z}_{i+1} = \mathbf{x}_0 - \frac{1}{L} \sum_{k=0}^i 2t_k \nabla f(\mathbf{x}_k)$ <p>Choose $t_{i+1} > 0$ s.t. $t_{i+1}^2 \leq T_{i+1} = \sum_{l=0}^{i+1} t_l$</p> $\mathbf{x}_{i+1} = \left(1 - \frac{t_{i+1}}{T_{i+1}}\right) \mathbf{y}_{i+1} + \frac{t_{i+1}}{T_{i+1}} \mathbf{z}_{i+1}$
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Clearly when $\theta_i = t_i$ for $i < N$, the primary iterates $\{\mathbf{y}_i\}_{i=0}^N$ and the intermediate secondary iterates $\{\mathbf{x}_i\}_{i=0}^{N-1}$ of GOGM and GOGM' are equivalent. Although illustrating two similar algorithms GOGM and GOGM' might seem redundant, presenting both formulations with lemmas 2 and 3 completes the story of generalized OGM here and in Section 5.

Using lemmas 2 and 3, the following theorem bounds the cost function decrease of the GOGM and GOGM' iterates.

Theorem 5 Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be $\mathcal{F}_L(\mathbb{R}^d)$ and let $\mathbf{y}_0, \dots, \mathbf{y}_N, \mathbf{x}_N \in \mathbb{R}^d$ be generated by Algorithm GOGM1 and GOGM2. Then for any $1 \leq i \leq N$,

$$f(\mathbf{y}_i) - f(\mathbf{x}_*) \leq \frac{LR^2}{4\Omega_{i-1}} \quad (30)$$

and

$$f(\mathbf{x}_N) - f(\mathbf{x}_*) \leq \frac{LR^2}{2\Omega_N}. \quad (31)$$

The iterates $\mathbf{y}_0, \dots, \mathbf{y}_N \in \mathbb{R}^d$ generated by Algorithm GOGM1' and GOGM2' also satisfy the bound (30) when $\theta_i = t_i$ for $i < N$.

Proof Using Lemma 3, FSFOM with the step coefficients \mathbf{h} in (27) of GOGM' satisfies

$$f(\mathbf{y}_{N+1}) - f(\mathbf{x}_*) \leq \mathcal{B}_{D'}(\mathbf{h}, N, L, R) = \frac{1}{2}LR^2\gamma = \frac{LR^2}{4T_N}, \quad (32)$$

where $\mathbf{y}_{N+1} = \mathbf{x}_N - \frac{1}{L}\nabla f(\mathbf{x}_N)$. Since the coefficients \mathbf{h} in (27) are recursive and do not depend on a given N , we can extend (32) for all iterations. By letting $\theta_i = t_i$ for $i < N$, the bound (32) also satisfies for the iterates $\{\mathbf{y}_i\}$ of GOGM, as in (30).

Using Lemma 2, FSFOM with the step \mathbf{h} in (24) of GOGM satisfies

$$f(\mathbf{x}_N) - f(\mathbf{x}_*) \leq \mathcal{B}_D(\mathbf{h}, N, L, R) = \frac{1}{2}LR^2\gamma = \frac{LR^2}{2\Omega_N}, \quad (33)$$

which is equivalent to (31).

GOGM and Thm. 5 reduce to OGM and its bounds (5) and (6), when $\theta_i^2 = \Omega_i$ for all i . Similar to general forms of FGM in [2, 18], the GOGM includes the choice $\theta_i = \begin{cases} \frac{i+a}{a}, & i < N, \\ \frac{\sqrt{2(N+a-1)}}{a}, & i = N \end{cases}$ for any $a \geq 2$, because such parameter θ_i satisfies the following conditions for GOGM:

$$\Omega_i - \theta_i^2 = \frac{(i+1)(i+2a)}{2a} - \frac{(i+a)^2}{a^2} = \frac{(a-2)i^2 + a(2a-3)}{2a^2} \geq 0, \quad (34)$$

for $i < N$, and $\Omega_N - \theta_N^2 = 2\Omega_{N-1} + \theta_N - \theta_N^2 \geq 2\theta_{N-1}^2 + \theta_N - \theta_N^2 = \theta_N \geq 0$. Similarly, the GOGM' includes the choice $t_i = \frac{i+a}{a}$ for any $a \geq 2$ as below, which we denote as OGM- a .

Corollary 1 Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be $\mathcal{F}_L(\mathbb{R}^d)$ and let $\mathbf{y}_0, \dots, \mathbf{y}_N \in \mathbb{R}^d$ be generated by Algorithm GOGM' with $t_i = \frac{i+a}{a}$ (OGM- a) for any $a \geq 2$. Then for any $1 \leq i \leq N$,

$$f(\mathbf{y}_i) - f(\mathbf{x}_*) \leq \frac{aLR^2}{2i(i+2a-1)}. \quad (35)$$

Proof Thm. 5 implies (35), since $T_i = \frac{(i+1)(i+2a)}{2a}$ and the condition $T_i - t_i^2 \geq 0$ satisfies in (34) for any $a \geq 2$.

The next section analyzes a convergence bound for the gradient of GOGM (and GOGM') using the gradient form of PEP. We use relaxations on PEP that are similar but slightly different from those of PEP for the cost function in this section. Then, we optimize the step coefficients with respect to the gradient form of PEP and propose an algorithm named OGM-OG that lies in the GOGM class.

5 Relaxation and optimization of the gradient form of PEP

5.1 Relaxation for the gradient form of PEP

To analyze a worst-case convergence bound on the gradient for FSFOM with a given \mathbf{h} , we consider the following gradient-form version of PEP that is similar to (P):

$$\begin{aligned} \mathcal{B}_{P''}(\mathbf{h}, N, d, L, R) := & \max_{\substack{f \in \mathcal{F}_L(\mathbb{R}^d), \\ \mathbf{x}_0, \dots, \mathbf{x}_N \in \mathbb{R}^d, \\ \mathbf{x}_* \in X_*(f)}} \min_{i \in \{0, \dots, N\}} \|\nabla f(\mathbf{x}_i)\|^2 \\ \text{s.t. } & \mathbf{x}_{i+1} = \mathbf{x}_i - \frac{1}{L} \sum_{k=0}^i h_{i+1,k} \nabla f(\mathbf{x}_k), \quad i = 0, \dots, N-1, \\ & \|\mathbf{x}_0 - \mathbf{x}_*\| \leq R. \end{aligned} \quad (P'')$$

Here, we used the smallest gradient norm squared among all iterates ($\min_{i \in \{0, \dots, N\}} \|\nabla f(\mathbf{x}_i)\|^2$) as a criteria, as considered in [23, Section 4.3]. We could instead consider the final gradient norm squared ($\|\nabla f(\mathbf{x}_N)\|^2$) as a criteria, but our proposed relaxation on (P'') in this section for such criteria did not provide any interesting result; we leave studying the gradient form of the tight PEP as future work.

As in [23], we replace $\min_{i \in \{0, \dots, N\}} \|\nabla f(\mathbf{x}_i)\|^2$ in (P'') by $L^2 \|\mathbf{x}_0 - \mathbf{x}_*\|^2 \alpha$ with the condition $\alpha \leq \frac{1}{L^2 \|\mathbf{x}_0 - \mathbf{x}_*\|^2} \|\nabla f(\mathbf{x}_i)\|^2 = \text{Tr}\{\mathbf{G}^\top (\mathbf{u}_i \mathbf{u}_i^\top) \mathbf{G}\}$ for all i . Then, we relax this reformulated (P'') similar to the relaxation from (P) to (P2) with the additional constraint $\text{Tr}\{\mathbf{G}^\top \mathbf{C}_N \mathbf{G}\} \leq \delta_N$ in (P1) as follows:

$$\begin{aligned} \mathcal{B}_{P2''}(\mathbf{h}, N, d, L, R) := & \max_{\substack{\mathbf{G} \in \mathbb{R}^{(N+1)d}, \\ \boldsymbol{\delta} \in \mathbb{R}^{N+1}, \\ \alpha \in \mathbb{R}}} L^2 R^2 \alpha \\ \text{s.t. } & \text{Tr}\{\mathbf{G}^\top \mathbf{A}_{i-1,i}(\mathbf{h}) \mathbf{G}\} \leq \delta_{i-1} - \delta_i, \quad i = 1, \dots, N, \\ & \text{Tr}\{\mathbf{G}^\top \mathbf{C}_N \mathbf{G}\} \leq \delta_N, \\ & \text{Tr}\{\mathbf{G}^\top \mathbf{D}_i(\mathbf{h}) \mathbf{G} + \nu \mathbf{u}_i^\top \mathbf{G}\} \leq -\delta_i, \quad i = 0, \dots, N, \\ & \text{Tr}\{\mathbf{G}^\top (-\mathbf{u}_i \mathbf{u}_i^\top) \mathbf{G}\} \leq -\alpha, \quad i = 0, \dots, N. \end{aligned} \quad (P2'')$$

Replacing $\max_{\mathbf{G}, \boldsymbol{\delta}} L^2 R^2 \alpha$ by $\min_{\mathbf{G}, \boldsymbol{\delta}} \{-\alpha\}$ for convenience, the Lagrangian of the corresponding constrained minimization problem with dual variables $\boldsymbol{\lambda} \in \mathbb{R}_+^N$, $\eta \in \mathbb{R}_+$, $\boldsymbol{\tau} \in \mathbb{R}_+^{N+1}$, and $\boldsymbol{\beta} = (\beta_0, \dots, \beta_N)^\top \in \mathbb{R}_+^{N+1}$ for the first, second, third, and fourth set of constraint inequalities of (P2'') respectively becomes

$$\begin{aligned} \mathcal{L}''(\mathbf{G}, \boldsymbol{\delta}, \boldsymbol{\lambda}, \eta, \boldsymbol{\tau}, \boldsymbol{\beta}; \mathbf{h}) = & -\alpha + \sum_{i=1}^N \lambda_i (\delta_i - \delta_{i-1}) - \eta \delta_N + \sum_{i=0}^N \tau_i \delta_i + \sum_{i=0}^N \beta_i \alpha \\ & + \text{Tr}\{\mathbf{G}^\top \mathbf{S}''(\mathbf{h}, \boldsymbol{\lambda}, \eta, \boldsymbol{\tau}, \boldsymbol{\beta}) \mathbf{G} + \nu \boldsymbol{\tau}^\top \mathbf{G}\}, \end{aligned} \quad (36)$$

where

$$\mathbf{S}''(\mathbf{h}, \boldsymbol{\lambda}, \eta, \boldsymbol{\tau}, \boldsymbol{\beta}) := \mathbf{S}(\mathbf{h}, \boldsymbol{\lambda}, \tau) + \frac{1}{2} \eta \mathbf{u}_N \mathbf{u}_N^\top - \sum_{i=0}^N \beta_i \mathbf{u}_i \mathbf{u}_i^\top. \quad (37)$$

Then similar to (D), we have the following upper bound on the PEP of the smallest gradient norm squared among all iterates that could be solved by a SDP solver:

$$\mathcal{B}_{D''}(\mathbf{h}, N, L, R) := \min_{\substack{(\boldsymbol{\lambda}, \eta, \boldsymbol{\tau}, \boldsymbol{\beta}) \in A'', \\ \gamma \in \mathbb{R}}} \left\{ \frac{1}{2} L^2 R^2 \gamma : \begin{pmatrix} \mathbf{S}''(\mathbf{h}, \boldsymbol{\lambda}, \eta, \boldsymbol{\tau}, \boldsymbol{\beta}) & \frac{1}{2} \boldsymbol{\tau} \\ \frac{1}{2} \boldsymbol{\tau}^\top & \frac{1}{2} \gamma \end{pmatrix} \succeq 0 \right\}, \quad (D'')$$

where

$$A'' := \left\{ (\boldsymbol{\lambda}, \eta, \boldsymbol{\tau}, \boldsymbol{\beta}) \in \mathbb{R}_+^{3N+3} : \begin{array}{l} \tau_0 = \lambda_1, \quad \lambda_N + \tau_N = \eta, \quad \sum_{i=0}^N \beta_i = 1, \\ \lambda_i - \lambda_{i+1} + \tau_i = 0, \quad i = 1, \dots, N-1 \end{array} \right\}. \quad (38)$$

The next two sections use a valid upper bound (D'') of (P'') for given step coefficients \mathbf{h} , providing an analytical solution to (D'') for the step coefficients \mathbf{h} of FGM and GOGM', superseding the use of a SDP solver.

5.2 A convergence bound for the gradient of FGM

FGM is equivalent to FSFOM with the step coefficients [11, Prop. 1]

$$h_{i+1,k} = \begin{cases} \frac{1}{t_{i+1}} \left(t_k - \sum_{j=k+1}^i h_{j,k} \right), & k = 0, \dots, i-1, \\ 1 + \frac{t_i-1}{t_{i+1}}, & k = i, \end{cases} \quad (39)$$

for t_i in (4), and the following lemma provides a feasible point of (D'') associated with the step coefficients (39) of FGM to provide a convergence bound for the gradient of FGM.

Lemma 4 *For the step coefficients (39), the following choice of variables:*

$$\lambda_i = t_{i-1}^2 \tau_0, \quad i = 1, \dots, N, \quad \tau_i = \begin{cases} \left(\frac{1}{2} \sum_{k=0}^N t_k^2 \right)^{-1}, & i = 0, \\ t_i \tau_0, & i = 1, \dots, N, \end{cases} \quad \gamma = \tau_0 \quad (40)$$

$$\eta = t_N^2 \tau_0, \quad \beta_i = \frac{1}{2} t_i^2 \tau_0, \quad i = 0, \dots, N, \quad (41)$$

is a feasible point of (D'') for t_i in (4).

Proof See Appendix.

Using Lemma 4, the following theorem bounds the gradient norm of the FGM iterates, proving for the first time an $O(1/N^{\frac{3}{2}})$ rate of decrease.

Theorem 6 *Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be $\mathcal{F}_L(\mathbb{R}^d)$ and let $\mathbf{y}_0, \dots, \mathbf{y}_N, \mathbf{x}_0, \dots, \mathbf{x}_N \in \mathbb{R}^d$ be generated by Algorithm FGM. Then for any $N \geq 1$,*

$$\min_{i \in \{0, \dots, N+1\}} \|\nabla f(\mathbf{y}_i)\| \leq \min_{i \in \{0, \dots, N\}} \|\nabla f(\mathbf{x}_i)\| \leq \frac{LR}{\sqrt{\sum_{k=0}^N t_k^2}} \leq \frac{2\sqrt{3}LR}{\sqrt{(N+1)(N^2+6N+12)}}, \quad (42)$$

where $\mathbf{y}_{N+1} = \mathbf{x}_N - \frac{1}{L} \nabla f(\mathbf{x}_N)$.

Proof Lemma 1 implies the first inequality in (42). Using Lemma 4, FSFOM with the step coefficients \mathbf{h} (39) of FGM satisfies

$$\min_{i \in \{0, \dots, N\}} \|\nabla f(\mathbf{x}_i)\|^2 \leq \mathcal{B}_{D''}(\mathbf{h}, N, L, R) = \frac{1}{2} L^2 R^2 \gamma = \frac{L^2 R^2}{\sum_{k=0}^N t_k^2}, \quad (43)$$

which is equivalent to (42) using $\sum_{k=0}^N t_k^2 \geq \sum_{k=0}^N \frac{(k+2)^2}{4} = \frac{(N+1)(2N^2+13N+24)}{24}$.

5.3 A convergence bound for the gradient of GOGM

To bound the gradient decrease of GOGM (and GOGM'), the following lemma illustrates one possible set of feasible points of (D'').

Lemma 5 *For the step coefficients (27), the following choice of variables:*

$$\lambda_i = T_{i-1} \tau_0, \quad i = 1, \dots, N, \quad \tau_i = \begin{cases} \left(\sum_{k=0}^N (T_k - t_k^2) \right)^{-1}, & i = 0, \\ t_i \tau_0, & i = 1, \dots, N, \end{cases} \quad \gamma = \frac{1}{2} \tau_0 \quad (44)$$

$$\eta = T_N \tau_0, \quad \beta_i = (T_i - t_i^2) \tau_0, \quad i = 0, \dots, N \quad (45)$$

is a feasible point of (D'') for any choice of t_i and T_i that satisfies (29) and for which there exists some i such that $t_i^2 < T_i$.

Proof See Appendix.

Using Lemma 5, the following theorem bounds the gradient norm for the iterates of GOGM and GOGM'.

Theorem 7 Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be $\mathcal{F}_L(\mathbb{R}^d)$ and let $\mathbf{y}_0, \dots, \mathbf{y}_N, \mathbf{x}_0, \dots, \mathbf{x}_N \in \mathbb{R}^d$ be generated by Algorithm GOGM'. Then for any $N \geq 1$,

$$\min_{i \in \{0, \dots, N+1\}} \|\nabla f(\mathbf{y}_i)\| \leq \min_{i \in \{0, \dots, N\}} \|\nabla f(\mathbf{x}_i)\| \leq \frac{LR}{2\sqrt{\sum_{k=0}^N (T_k - t_k^2)}}, \quad (46)$$

where $\mathbf{y}_{N+1} = \mathbf{x}_N - \frac{1}{L}\nabla f(\mathbf{x}_N)$. The bound (46) can be generalized to the intermediate iterates $\{\mathbf{x}_i\}_{i=0}^{N-1}$ and $\{\mathbf{y}_i\}_{i=0}^N$ of both GOGM and GOGM' when $\theta_i = t_i$ (for $i < N$).

Proof Lemma 1 implies the first inequality in (46). Using Lemma 5, FSFOM with the step coefficients \mathbf{h} (27) of GOGM' satisfies

$$\min_{i \in \{0, \dots, N\}} \|\nabla f(\mathbf{x}_i)\|^2 \leq \mathcal{B}_{D''}(\mathbf{h}, N, L, R) = \frac{1}{2}L^2R^2\gamma = \frac{L^2R^2}{4\sum_{k=0}^N (T_k - t_k^2)},$$

which implies (46). Since the iterates of GOGM' are recursive and do not depend on a given N , the bound (46) easily generalizes to the intermediate iterates of GOGM' (and GOGM when $\theta_i = t_i$).

5.4 Optimizing step coefficients over the gradient form of PEP

In search of a FSFOM that decreases the gradient norm the fastest, we optimize the step coefficients in terms of the gradient form of the relaxed (D'') by solving the following problem:

$$\hat{\mathbf{h}}_{D''} := \arg \min_{\mathbf{h} \in \mathbb{R}^{N(N+1)/2}} \mathcal{B}_{D''}(\mathbf{h}, N, L, R). \quad (\text{HD}'')$$

The problem (HD'') is bilinear, similar to (HD), and a convex relaxation technique [5, Thm. 3] makes this problem solvable using numerical methods.

We solved (HD'') for many choices of N using a numerical SDP solver [3, 8], and observed that the following choice of t_i :

$$t_i = \begin{cases} 1, & i = 0, \\ \frac{1 + \sqrt{1 + 4t_{i-1}^2}}{2}, & i = 1, \dots, \lfloor \frac{N}{2} \rfloor - 1, \\ \frac{N-i+1}{2}, & i = \lfloor \frac{N}{2} \rfloor, \dots, N, \end{cases} \quad (47)$$

makes the feasible point in Lemma 5 optimal for the problem (HD''). Based on that numerical evidence, we conjecture that $\hat{\mathbf{h}}_{D''}$ in (HD'') corresponds to the step coefficients (27) with the parameter t_i (47). It is interesting that the t_i factors in (47) start decreasing after $i = \lfloor \frac{N}{2} \rfloor - 1$, whereas the usual t_i in (4) and $t_i = \frac{i+a}{a}$ for any $a \geq 2$ increase with i indefinitely.

In addition, we found numerically that minimizing the gradient bound (46) of GOGM', i.e., solving the following constrained quadratic problem:

$$\max_{\{t_i\}} \sum_{k=0}^N \left(\sum_{l=0}^k t_l - t_k^2 \right) \quad \text{s.t.} \quad t_i \text{ satisfies (29) for all } i, \quad (48)$$

is equivalent to solving the problem (HD''). In other words, the solution of (48) numerically appears equivalent to (47), the (conjectured) solution of (HD''). Interestingly, the unconstrained maximizer of the cost function of (48) is $t_i = \frac{N-i+1}{2}$, and this partially appears in the constrained maximizer (47) for $\lfloor \frac{N}{2} \rfloor \leq i \leq N$.

We denote the resulting GOGM' with (47) as OGM-OG (OG for optimized over gradient). The following theorem bounds the cost function and gradient norm of the OGM-OG iterates.

Theorem 8 Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be $\mathcal{F}_L(\mathbb{R}^d)$ and let $\mathbf{y}_0, \dots, \mathbf{y}_N, \mathbf{x}_0, \dots, \mathbf{x}_N \in \mathbb{R}^d$ be generated by Algorithm OGM-OG. Then,

$$f(\mathbf{y}_{N+1}) - f(\mathbf{x}_*) \leq \frac{2LR^2}{(N+2)^2}, \quad (49)$$

and

$$\min_{i \in \{0, \dots, N+1\}} \|\nabla f(\mathbf{y}_i)\| \leq \min_{i \in \{0, \dots, N\}} \|\nabla f(\mathbf{x}_i)\| \leq \frac{\sqrt{6}LR}{N\sqrt{N+1}}, \quad (50)$$

where $\mathbf{y}_{N+1} = \mathbf{x}_N - \frac{1}{L}\nabla f(\mathbf{x}_N)$.

Proof OGM-OG is an instance of GOGM' and thus Thm. 5 implies that OGM-OG satisfies

$$f(\mathbf{y}_{N+1}) - f(\mathbf{x}_*) \leq \frac{LR^2}{4T_N},$$

which is equivalent to (49), since

$$\begin{aligned} T_N &= T_m + \sum_{l=m+1}^N t_l = t_m^2 + \frac{(N-m)(N-m+1)}{4} \\ &\geq \frac{(N+3)^2 + N(N+2)}{16} = \frac{2N^2 + 8N + 9}{16} \end{aligned}$$

for $m = \lfloor \frac{N}{2} \rfloor$, using $m \geq \frac{N-1}{2}$, $N-m \geq \frac{N}{2}$, and $t_m \geq \frac{m+2}{2} \geq \frac{N+3}{4}$ in (4).

Thm. 7 implies (50), using the above inequalities for m , the equality $t_i^2 = T_i$ for $i \leq m$, and

$$\begin{aligned} &\sum_{k=m+1}^N (T_k - t_k^2) \\ &= \sum_{k=m+1}^N \left(t_m^2 + \sum_{l=m+1}^k t_l - t_k^2 \right) \\ &= (N-m)t_m^2 + \sum_{k=m+1}^N \left(\sum_{l=m+1}^k \frac{N-l+1}{2} - \left(\frac{N-k+1}{2} \right)^2 \right) \\ &= (N-m)t_m^2 + \sum_{k'=1}^{N-m} \left(\sum_{l'=1}^{k'} \frac{N-l'-m+1}{2} - \left(\frac{N-k'-m+1}{2} \right)^2 \right) \\ &= (N-m)t_m^2 \\ &\quad + \sum_{k=1}^{N-m} \left(\frac{2(N-m+1)k - k(k+1)}{4} - \frac{(N-m+1)^2 - 2(N-m+1)k + k^2}{4} \right) \\ &= (N-m)t_m^2 + \sum_{k=1}^{N-m} \left(-\frac{k^2}{2} + (N-m+3/4)k - \frac{(N-m+1)^2}{4} \right) \\ &= (N-m)t_m^2 - \frac{(N-m)(N-m+1/2)(N-m+1)}{6} \\ &\quad + \frac{(N-m)(N-m+3/4)(N-m+1)}{2} - \frac{(N-m)(N-m+1)^2}{4} \\ &\geq \frac{(N-m)(m+2)^2}{4} + \frac{(N-m)^2(N-m+1)}{3} - \frac{(N-m)(N-m+1)^2}{4} \\ &\geq \frac{(N-m)^2(N-m+1)}{3} \\ &\geq \frac{1}{24}N^2(N+1). \end{aligned}$$

The gradient bound (50) of OGM-OG is asymptotically $\sqrt{2}$ -times smaller than that of FGM in Thm. 6 and $\frac{2\sqrt{6}}{3} \approx 1.63$ -times smaller than that of OGM-H in Thm. 4. Regarding the cost function decrease, the bound (49) of OGM-OG is asymptotically the same as the bound (3) of FGM, and both are twice larger than the bounds (5) and (6) of OGM.

5.5 Decreasing the gradient norm with rate $O(1/N^{\frac{3}{2}})$ using GOGM without selecting N in advance

Although OGM-OG satisfies a small gradient bound with a rate $O(1/N^{\frac{3}{2}})$, OGM-OG (and OGM-H) must select N in advance, unlike FGM. Using Thm. 7, the following corollary shows that OGM- a with $a > 2$ can decrease the gradient with a rate $O(1/N^{\frac{3}{2}})$ without selecting N in advance. (Cor. 1 showed that OGM- a algorithm with $a \geq 2$ can decrease the cost function with an optimal rate $O(1/N^2)$.)

Corollary 2 *Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be $\mathcal{F}_L(\mathbb{R}^d)$ and let $\mathbf{y}_0, \dots, \mathbf{y}_N, \mathbf{x}_0, \dots, \mathbf{x}_N \in \mathbb{R}^d$ be generated by Algorithm GOGM' with $t_i = \frac{i+a}{a}$ (OGM- a) for any $a \geq 2$. Then for $N \geq 1$,*

$$\begin{aligned} \min_{i \in \{0, \dots, N+1\}} \|\nabla f(\mathbf{y}_i)\| &\leq \min_{i \in \{0, \dots, N\}} \|\nabla f(\mathbf{x}_i)\| \\ &\leq \frac{a\sqrt{6}LR}{2\sqrt{N(N+1)((a-2)N + (3a^2 - 4a - 2))}}, \end{aligned} \quad (51)$$

where $\mathbf{y}_{N+1} = \mathbf{x}_N - \frac{1}{L}\nabla f(\mathbf{x}_N)$.

Proof Using $T_i = \frac{(i+1)(i+2a)}{2a}$ and (34), Thm. 7 implies (51) since

$$\begin{aligned} \sum_{k=0}^N (T_k - t_k^2) &= \sum_{k=0}^N \frac{(a-2)k^2 + a(2a-3)k}{2a^2} \\ &= \frac{N(N+1)((a-2)N + (3a^2 - 4a - 2))}{6a^2}. \end{aligned}$$

OGM- a for any $a > 2$ has a gradient bound (51) that is about $\frac{a}{2\sqrt{a-2}}$ -times larger than the bound (50) of OGM-OG. This constant factor minimizes to $\sqrt{2}$ when $a = 4$, and this OGM- $a = 4$ has a gradient bound that is asymptotically equivalent to the bound (42) of FGM. Therefore, when one does not want to select N in advance, both FGM and OGM- $a=4$ will be useful for decreasing the gradient with a rate $O(1/N^{\frac{3}{2}})$.

6 Discussion

Table 1 summarizes the asymptotic rate of analytical convergence bounds of all algorithms presented in this paper. As discussed, OGM and OGM-OG have the best convergence bounds for the cost function and gradient decrease respectively in Table 1. However, since OGM has a slow convergence rate for the gradient decrease, other algorithms such as FGM, OGM-H, OGM-OG, and OGM- a that satisfy both the optimal rate $O(1/N^2)$ for the function decrease and a fast rate $O(1/N^{\frac{3}{2}})$ for the gradient decrease could be preferable over OGM when one is interested in the gradient decrease as well as the function decrease, particularly when solving dual problems. In addition, when one does not want to choose N in advance, FGM and OGM- a could be preferable. For further acceleration of the gradient norm decrease, a regularization technique in [19] that minimizes the gradient norm with rate $O(1/N^2)$ up to a logarithmic factor could be used. However, its (dynamic) step coefficients require knowing R in advance which is undesirable in practice. Whether there exists FSFOM satisfying such rate is an open question.

Since many bounds presented in Table 1 are not guaranteed to be tight, we used the code in Taylor *et al.* [23] to compare tight (numerical) bounds for $N = 1, 2, 4, 10, 20, 30, 40, 48, 50$. Note that this numerical bound is guaranteed to be tight when the large-scale condition “ $d \geq N + 2$ ” is satisfied [23, Thm. 5], and we assume this condition hereafter. Tables 2 and 3 provide tight bounds for the decrease of the cost

Algorithm	Asymptotic convergence rate bound		Require selecting N in advance
	Cost function	Gradient norm	
GM	$\frac{1}{4}N^{-1}$	$\sqrt{2}N^{-1}$	No
FGM	$2N^{-2}$	$2\sqrt{3}N^{-\frac{3}{2}}$	No
OGM	N^{-2}	$\sqrt{2}N^{-1}$	No
OGM-H	$4N^{-2}$	$4N^{-\frac{3}{2}}$	Yes
OGM-OG	$2N^{-2}$	$\sqrt{6}N^{-\frac{3}{2}}$	Yes
OGM- a ($a > 2$)	$\frac{a}{2}N^{-2}$	$\frac{a\sqrt{6}}{2\sqrt{a-2}}N^{-\frac{3}{2}}$	No
OGM- $a=4$	$2N^{-2}$	$2\sqrt{3}N^{-\frac{3}{2}}$	

Table 1 Asymptotic convergence rate bounds on the cost function $\frac{1}{LR^2}(f(\mathbf{x}_N) - f(\mathbf{x}_*))$ and the gradient norm $\min_{i \in \{0, \dots, N\}} \frac{1}{LR} \|\nabla f(\mathbf{x}_i)\|$ of GM, FGM, OGM, OGM-H, OGM-OG, and OGM- a . (The cost function bound for OGM-H in the table corresponds to the bound for OGM for total $\lfloor N/2 \rfloor$ iterations, while the final iterate of OGM-H has a lower bound.)

function $f(\mathbf{x}_N) - f(\mathbf{x}_*)$ and the gradient norm decrease $\min_{i \in \{0, \dots, N\}} \|\nabla f(\mathbf{x}_i)\|$ respectively.³ Although most of the bounds in Table 1 are not guaranteed be tight, the convergence rate behaviors in Table 1 are similar to those in Tables 2 and 3, except that the gradient bounds of OGM-H in Table 1 are relatively looser than those of OGM-H in Table 3.

N	GM	FGM	OGM	OGM-H	OGM-OG	OGM- $a=4$
1	6.00	6.00	8.00	6.00	7.33	6.48
2	10.00	11.13	16.16	12.00	13.20	15.11
4	18.00	24.66	39.09	24.16	28.56	32.33
10	42.00	90.69	159.07	73.80	99.89	106.44
20	82.00	283.55	525.09	199.07	310.39	308.90
30	122.00	578.58	1095.62	376.37	604.93	610.95
40	162.00	975.10	1869.20	605.09	1009.91	1012.83
48	194.00	1365.15	2633.77	824.87	1406.69	1406.29
50	202.00	1472.76	2845.12	884.91	1515.98	1514.64
Empi. $O(\cdot)$	$N^{-0.99}$	$N^{-1.86}$	$N^{-1.89}$	$N^{-1.72}$	$N^{-1.83}$	$N^{-1.82}$
Known $O(\cdot)$	N^{-1}	N^{-2}	N^{-2}	N^{-2}	N^{-2}	N^{-2}

Table 2 Tight worst-case convergence bounds on the cost function $\frac{LR^2}{f(\mathbf{x}_N) - f(\mathbf{x}_*)}$ of GM, FGM, OGM, OGM-H, OGM-OG, and OGM- $a=4$. We computed empirical rates by assuming that the bounds follow the form bN^{-c} with constants b and c , and then by estimating c from points $N = 48, 50$. Note that the corresponding empirical rates are underestimated due to its simple modeling of bounds.

To be clear, Section 5 and Tables 1, 3 have focused on analyzing the *smallest* gradient norm among all iterates using the gradient form of PEP, whereas the gradient analysis in Section 3.2 considers the *final* gradient in addition to the smallest gradient among all iterates. As mentioned before, we have not yet found a relaxation on the *final* gradient form of the PEP that provides as interesting (comparable) results as for the relaxation on the *smallest* gradient form of the PEP (P'') in Section 5. To complete comparisons on the gradient bounds, Table 4 uses the code provided by Taylor *et al.* [23] to compare tight (numerical) bounds on the *final* gradient of the FSFOM presented in this paper.

Interestingly in Table 4, FGM and OGM- $a=4$ have slow $O(1/N)$ bounds on the final gradient, unlike OGM-H and OGM-OG roughly having $O(1/N^{\frac{3}{2}})$ bounds for both the smallest and final gradients. (Thm. 4 has shown that the final gradient of OGM-H satisfies a rate $O(1/N^{\frac{3}{2}})$, but this is unknown yet for OGM-OG, which we leave as future work.) Considering the results in Tables 3 and 4, it would be interesting to develop FSFOM that has $O(1/N^{\frac{3}{2}})$ bounds for the final gradient decrease that are lower than those of OGM-H and OGM-OG, possibly without requiring to choose N in advance.

³ Numerical tight bounds on two sequences $\{\mathbf{y}_i\}$ and $\{\mathbf{x}_i\}$ are observed similar in big-O sense. In particular, tight bounds for the cost function $f(\mathbf{x}_N) - f(\mathbf{x}_*)$ are numerically found similar but smaller than those of $f(\mathbf{y}_N) - f(\mathbf{x}_*)$ for the algorithms in Table 2 for $N = 1, 2, 4, 10, 20, 30, 40, 48, 50$. On the other hand, we numerically found that tight bounds for the gradient norm decrease $\min_{i \in \{0, \dots, N\}} \|\nabla f(\mathbf{x}_i)\|$ is similar in big-O sense but larger than those for $\min_{i \in \{0, \dots, N\}} \|\nabla f(\mathbf{y}_i)\|$; we show such tight bounds only for the sequence $\{\mathbf{x}_i\}$ for simplicity here.

N	GM	FGM	OGM	OGM-H	OGM-OG	OGM- $a=4$
1	2.00	2.00	2.00	2.00	2.33	1.80
2	3.00	3.28	2.84	3.50	3.67	3.29
4	5.00	5.85	4.42	6.36	6.78	5.71
10	11.00	13.82	8.92	17.20	18.87	15.29
20	21.00	32.83	16.20	40.87	45.39	35.25
30	31.00	56.36	23.41	70.16	78.56	59.38
40	41.00	83.64	30.57	103.80	116.88	87.09
48	49.00	107.84	36.29	133.77	150.97	111.56
50	51.00	114.20	37.72	141.55	160.00	117.98
Empi. $O(\cdot)$	$N^{-0.98}$	$N^{-1.40}$	$N^{-0.95}$	$N^{-1.38}$	$N^{-1.42}$	$N^{-1.37}$
Known $O(\cdot)$	N^{-1}	$N^{-\frac{3}{2}}$	N^{-1}	$N^{-\frac{3}{2}}$	$N^{-\frac{3}{2}}$	$N^{-\frac{3}{2}}$

Table 3 Tight worst-case convergence bounds on the gradient norm $\frac{LR}{\min_{i \in \{0, \dots, N\}} \|\nabla f(\mathbf{x}_i)\|}$ of GM, FGM, OGM, OGM-H, OGM-OG, and OGM- $a=4$. Empirical rates were computed as described in Table 2.

N	GM	FGM	OGM	OGM-H	OGM-OG	OGM- $a=4$
1	2.00	2.00	2.00	2.00	2.33	1.80
2	3.00	3.28	2.84	3.50	3.67	3.29
4	5.00	5.85	4.42	6.36	6.78	5.10
10	11.00	8.22	8.92	17.20	18.85	8.67
20	21.00	13.15	16.20	40.87	44.38	13.81
30	31.00	18.17	23.41	70.16	74.09	18.84
40	41.00	23.20	30.57	103.80	106.99	23.85
48	49.00	27.23	36.29	133.77	135.25	27.85
50	51.00	28.23	37.72	141.55	142.55	28.85
Empi. $O(\cdot)$	$N^{-0.98}$	$N^{-0.88}$	$N^{-0.95}$	$N^{-1.38}$	$N^{-1.29}$	$N^{-0.86}$
Known $O(\cdot)$	N^{-1}	N^{-1}	N^{-1}	$N^{-\frac{3}{2}}$	N^{-1}	N^{-1}

Table 4 Tight worst-case convergence bounds on the final gradient norm $\frac{LR}{\|\nabla f(\mathbf{x}_N)\|}$ of GM, FGM, OGM, OGM-H, OGM-OG, and OGM- $a=4$. Empirical rates were computed as described in Table 2. The known bounds of OGM-OG and OGM- $a=4$ are derived based on Section 3.2, where the empirical bounds of OGM-OG are comparable to the bounds of OGM-H with known rate $O(1/N^{\frac{3}{2}})$.

Because OGM is optimal in terms of the function decrease when $d \geq N + 1$ [4], one might hope that the OGM-OG achieves the optimal complexity in terms of the gradient decrease, since OGM-OG is also derived by optimizing the step coefficients over the gradient form of relaxed PEP. However, the OGM-OG is apparently not optimal as explained next.

Taylor *et al.* [23] numerically studied an optimal fixed-step GM using their tight PEP in terms of both the cost function and gradient decrease. In other words, they searched an optimal step h of GM:

$$\mathbf{x}_{i+1} = \mathbf{x}_i - \frac{h}{L} \nabla f(\mathbf{x}_i)$$

for $i = 0, \dots, N - 1$ and a given N with respect to $f(\mathbf{x}_N) - f(\mathbf{x}_*)$ and $\|\nabla f(\mathbf{x}_N)\|$. In the special case of $N = 1$, Taylor *et al.* [23] numerically conjectured that the step size $h = 1.5$ is optimal in terms of the cost function decrease. The corresponding GM is equivalent to OGM for $N = 1$, and this (numerically) implies the optimality of OGM for $N = 1$. They also numerically conjectured that the optimal step size of GM for $N = 1$ in terms of the gradient decrease is $h = \sqrt{2}$ with a bound

$$\|\nabla f(\mathbf{x}_1)\| \leq \frac{LR}{\sqrt{2} + 1} \approx \frac{LR}{2.41}. \quad (52)$$

However, OGM-OG for $N = 1$ reduces to GM with $h = \frac{4}{3} \approx 1.33$ with a bound $\frac{LR}{2.33}$ in Tables 3 and 4, implying that OGM-OG is not optimal even for $N = 1$ based on the numerical evidence in [23].

This analysis for $N = 1$ illustrates that there is still room for improvement in accelerating first-order methods in terms of gradients, which we leave as future work possibly with a tighter relaxation on the gradient form of PEP. In addition, it would be interesting to study the optimal complexity for the gradient decrease of first-order methods similar to [4], and develop FSFOM that achieves such

complexity. Nevertheless, the OGM-OG is the best known FSFOM for decreasing the gradient norm among the class FSFOM (without using a regularization technique in [19]), and will be useful when decreasing the gradient is key.

7 Conclusion

We generalized the formulation of OGM and analyzed its convergence bounds on the function value and gradient, using the cost function form and the gradient form of relaxed PEP. We then proposed OGM-OG by optimizing the step coefficients of FSFOM using a relaxed PEP with respect to the gradient, similar to the development of the (optimal) OGM. To the best of our knowledge, the convergence bound on the gradient of the OGM-OG is the best known analytical worst-case bound for decreasing the gradient norm among the class FSFOM.

However, this OGM-OG is not optimal for decreasing the gradient norm, and further accelerating FSFOM in terms of the gradient possibly with a tight relaxation on the gradient form of PEP is an interesting research direction. On the other hand, deriving a lower bound for the gradient norm of first-order methods, similar to that for the function decrease [4, 17], will be useful. Nonetheless, the proposed OGM-OG (and OGM- a) may be useful when one finds minimizing gradients important as well as decreasing the cost function, particularly in dual problems. In addition, we used the proposed gradient form of PEP to show that FGM decreases the (smallest) gradient with a rate $O(1/N^{\frac{3}{2}})$, implying that FGM is comparable in a big-O sense to OGM-H, OGM-OG and OGM- a for the gradient decrease.

Our analysis only considers unconstrained smooth convex minimization, and extending such gradient norm convergence analysis to constrained problems or nonsmooth composite convex problems would be a natural direction to pursue, which is studied for FGM by the authors [10]. In addition, it would be interesting to extend the analyses on the general form of FGM in [1, 2, 21] to our generalized OGM.

8 Appendix

8.1 Proof of Thm. 3

Due to (11), we have

$$\max_{f \in \mathcal{F}_L(\mathbb{R}^d)} \min_{i \in \{0, \dots, N\}} \|\nabla f(\mathbf{x}_i)\| \leq \max_{f \in \mathcal{F}_L(\mathbb{R}^d)} \|\nabla f(\mathbf{x}_N)\| \leq \frac{LR}{\theta_N},$$

and the rest of the proof shows that the following inequality holds

$$\begin{aligned} \frac{LR}{\theta_N} &= \min_{i \in \{0, \dots, N\}} \|\nabla \phi(\mathbf{x}_i)\| = \|\nabla \phi(\mathbf{x}_N)\| \\ &\leq \max_{f \in \mathcal{F}_L(\mathbb{R}^d)} \min_{i \in \{0, \dots, N\}} \|\nabla f(\mathbf{x}_i)\| \leq \max_{f \in \mathcal{F}_L(\mathbb{R}^d)} \|\nabla f(\mathbf{x}_N)\|, \end{aligned} \quad (53)$$

which then implies (16) with $\theta_N \geq \frac{N+1}{\sqrt{2}}$ (7).

Starting from $\mathbf{x}_0 = R\boldsymbol{\nu}$, where $\boldsymbol{\nu}$ is a unit vector, we first use induction to show that the following iterates:

$$\mathbf{x}_i = (-1)^i \frac{1}{\theta_i} R\boldsymbol{\nu}, \quad i = 0, \dots, N, \quad (54)$$

correspond to the iterates of OGM applied to $\phi(\mathbf{x})$. We use [11, Prop. 4] that the sequence generated by OGM is identical to the sequence generated by FSFOM with

$$h_{i+1,k} = \begin{cases} \frac{\theta_i-1}{\theta_{i+1}} h_{i,k}, & k = 0, \dots, i-2, \\ \frac{\theta_i-1}{\theta_{i+1}} (h_{i,i-1} - 1), & k = i-1, \\ 1 + \frac{2\theta_i-1}{\theta_{i+1}}, & k = i, \end{cases} \quad (55)$$

for $i = 0, \dots, N-1$.

Assuming that (54) holds for $i < N$, we have

$$\begin{aligned}
\mathbf{x}_{i+1} &= \mathbf{x}_i - \frac{1}{L} \sum_{k=0}^i h_{i+1,k} \nabla \phi(\mathbf{x}_k) \\
&= \mathbf{x}_i - \frac{1}{L} \left(1 + \frac{2\theta_i - 1}{\theta_{i+1}} \right) \nabla \phi(\mathbf{x}_i) - \frac{1}{L} \sum_{k=0}^{i-1} \frac{\theta_i - 1}{\theta_{i+1}} h_{i,k} \nabla \phi(\mathbf{x}_k) + \frac{1}{L} \frac{\theta_i - 1}{\theta_{i+1}} \nabla \phi(\mathbf{x}_{i-1}) \\
&= \frac{1 - 2\theta_i}{\theta_{i+1}} \mathbf{x}_i + \frac{\theta_i - 1}{\theta_{i+1}} (\mathbf{x}_i - \mathbf{x}_{i-1}) + \frac{\theta_i - 1}{\theta_{i+1}} \mathbf{x}_{i-1} \\
&= -\frac{\theta_i}{\theta_{i+1}} \mathbf{x}_i = (-1)^{i+1} \frac{1}{\theta_{i+1}} R\boldsymbol{\nu},
\end{aligned}$$

using (55) and $\nabla \phi(\mathbf{x}) = L\mathbf{x}$. Therefore, after N iterations of OGM we have:

$$\min_{i \in \{0, \dots, N\}} \|\nabla \phi(\mathbf{x}_i)\| = \|\nabla \phi(\mathbf{x}_N)\| = \left\| \nabla \phi \left((-1)^N \frac{1}{\theta_N} R\boldsymbol{\nu} \right) \right\| = \frac{LR}{\theta_N}, \quad (56)$$

which is equivalent to (53). The first equality of (56) holds since OGM monotonically decreases the gradient norm of $\phi(\mathbf{x})$, i.e. $\|\nabla \phi(\mathbf{x}_i)\| = \frac{LR}{\theta_i}$ monotonically decreases as i increases.

8.2 Proof of Lemma 2

It is obvious that $(\boldsymbol{\lambda}, \boldsymbol{\tau})$ in (25) is in Λ (21). Using (18) and (20), we have

$$\mathbf{S}(\mathbf{h}, \boldsymbol{\lambda}, \boldsymbol{\tau}) = \begin{cases} \frac{1}{2} \left((\lambda_i + \tau_i) h_{i,k} + \tau_i \sum_{j=k+1}^{i-1} h_{j,k} \right), & i = 2, \dots, N, \quad k = 0, \dots, i-2, \\ \frac{1}{2} ((\lambda_i + \tau_i) h_{i,k} - \lambda_i), & i = 1, \dots, N, \quad k = i-1, \\ \lambda_{i+1}, & i = 0, \dots, N-1, \quad k = i, \\ \frac{1}{2}, & i = N, \quad k = i, \end{cases} \quad (57)$$

and inserting (24) and (25), yields

$$\begin{aligned}
&\mathbf{S}(\mathbf{h}, \boldsymbol{\lambda}, \boldsymbol{\tau}) \\
&= \begin{cases} \frac{1}{2} \left(\Omega_i \tau_0 \frac{\theta_i}{\Omega_i} \left(2\theta_k - \sum_{j=k+1}^{i-1} h_{j,k} \right) + \theta_i \tau_0 \sum_{j=k+1}^{i-1} h_{j,k} \right), & i = 2, \dots, N-1, \quad k = 0, \dots, i-2, \\ \frac{1}{2} \left(\frac{\Omega_N}{2} \tau_0 \frac{\theta_N}{\Omega_N} \left(2\theta_k - \sum_{j=k+1}^{N-1} h_{j,k} \right) + \frac{\theta_N}{2} \tau_0 \sum_{j=k+1}^{N-1} h_{j,k} \right), & i = N, \quad k = 0, \dots, i-2, \\ \frac{1}{2} \left(\Omega_i \tau_0 \left(1 + \frac{(2\theta_{i-1}-1)\theta_i}{\Omega_i} \right) - \Omega_{i-1} \tau_0 \right), & i = 1, \dots, N-1, \quad k = i-1, \\ \frac{1}{2} \left(\frac{\Omega_N}{2} \tau_0 \left(1 + \frac{(2\theta_{N-1}-1)\theta_N}{\Omega_N} \right) - \Omega_{N-1} \tau_0 \right), & i = N, \quad k = i-1, \\ \Omega_i \tau_0, & i = 0, \dots, N-1, \quad k = i, \\ \frac{\Omega_N}{4} \tau_0, & i = N, \quad k = i, \end{cases} \\
&= \begin{cases} \theta_i \theta_k \tau_0, & i = 1, \dots, N-1, \quad k = 0, \dots, i-1, \\ \frac{\theta_N \theta_k}{2} \tau_0, & i = N, \quad k = 0, \dots, i-1, \\ \Omega_i \tau_0, & i = 1, \dots, N-1, \quad k = i, \\ \frac{\Omega_N}{4} \tau_0, & i = N, \quad k = i, \end{cases}
\end{aligned}$$

for θ_i and Ω_i in (26). Then, using (25) and (26), we show the feasibility condition of (D):

$$\begin{pmatrix} \mathbf{S}(\mathbf{h}, \boldsymbol{\lambda}, \boldsymbol{\tau}) & \frac{1}{2} \boldsymbol{\tau} \\ \frac{1}{2} \boldsymbol{\tau}^\top & \frac{1}{2} \gamma \end{pmatrix} = \left(\text{diag}\{\boldsymbol{\Omega} - \boldsymbol{\theta}^2\} + \boldsymbol{\theta} \boldsymbol{\theta}^\top \right) \boldsymbol{\tau}_0 \succeq 0,$$

where $\boldsymbol{\theta} = (\theta_0, \dots, \theta_{N-1}, \frac{\theta_N}{2}, \frac{1}{2})^\top$ and $\boldsymbol{\Omega} = (\Omega_0, \dots, \Omega_{N-1}, \frac{\Omega_N}{4}, \frac{1}{4})^\top$.

8.3 Proof of Lemma 3

It is obvious that $(\boldsymbol{\lambda}, \boldsymbol{\tau})$ in (28) is in Λ (21). Inserting (27) and (28) to (57) yields

$$\begin{aligned} & \mathbf{S}(\mathbf{h}, \boldsymbol{\lambda}, \boldsymbol{\tau}) + \frac{1}{2} \mathbf{u}_N \mathbf{u}_N^\top \\ &= \begin{cases} \frac{1}{2} \left(T_i \tau_0 \frac{t_i}{T_i} \left(2t_k - \sum_{j=k+1}^{i-1} h_{j,k} \right) + t_i \tau_0 \sum_{j=k+1}^{i-1} h_{j,k} \right), & i = 2, \dots, N-1, k = 0, \dots, i-2, \\ \frac{1}{2} \left(T_i \tau_0 \left(1 + \frac{(2t_{i-1}-1)t_i}{T_i} \right) - T_{i-1} \tau_0 \right), & i = 1, \dots, N, k = i-1, \\ T_i \tau_0, & i = 0, \dots, N, k = i \end{cases} \\ &= \begin{cases} t_i t_k \tau_0, & i = 1, \dots, N, k = 0, \dots, i-1, \\ T_i \tau_0, & i = 1, \dots, N, k = i, \end{cases} \end{aligned}$$

for t_i and T_i in (29). Then, using (28) and (29), we show the feasibility condition of (D'):

$$\begin{pmatrix} \mathbf{S}'(\mathbf{h}, \boldsymbol{\lambda}, \boldsymbol{\tau}) & \frac{1}{2} \boldsymbol{\tau} \\ \frac{1}{2} \boldsymbol{\tau}^\top & \frac{1}{2} \gamma \end{pmatrix} = (\text{diag}\{\mathbf{T} - \mathbf{t}^2\} + \mathbf{t} \mathbf{t}^\top) \tau_0 \succeq 0,$$

where $\mathbf{t} = (t_0, \dots, t_N, \frac{1}{2})^\top$ and $\mathbf{T} = (T_0, \dots, T_N, \frac{1}{4})^\top$.

8.4 Proof of Prop. 1

The proof consists of three propositions and they follow the derivations in [11, Prop. 3, 4 and 5] respectively. Note that this proof is independent of the choice of θ_i and Ω_i .

Proposition 2 *The step coefficient (24) satisfies the following recursive relationship*

$$h_{i+1,k} = \begin{cases} \frac{(\Omega_i - \theta_i) \theta_{i+1}}{\theta_i \Omega_{i+1}} h_{i,k}, & k = 0, \dots, i-2, \\ \frac{(\Omega_i - \theta_i) \theta_{i+1}}{\theta_i \Omega_{i+1}} (h_{i,i-1} - 1), & k = i-1, \\ 1 + \frac{(2\theta_i - 1) \theta_{i+1}}{\Omega_{i+1}}, & k = i, \end{cases} \quad (58)$$

for $i = 0, \dots, N-1$.

Proof We use the notation $h'_{i,k}$ for the coefficients (24) to distinguish from (58). It is obvious that $h'_{i+1,i} = h_{i+1,i}$, $i = 0, \dots, N-1$, and we clearly have

$$\begin{aligned} h'_{i+1,i-1} &= \frac{\theta_{i+1}}{\Omega_{i+1}} (2\theta_{i-1} - h'_{i,i-1}) = \frac{\theta_{i+1}}{\Omega_{i+1}} \left(2\theta_{i-1} - \left(1 + \frac{(2\theta_{i-1} - 1) \theta_i}{\Omega_i} \right) \right) \\ &= \frac{(2\theta_{i-1} - 1)(\Omega_i - \theta_i) \theta_{i+1}}{\Omega_i \Omega_{i+1}} = \frac{(\Omega_i - \theta_i) \theta_{i+1}}{\theta_i \Omega_{i+1}} (h_{i,i-1} - 1) = h_{i+1,i-1}. \end{aligned}$$

We next use induction by assuming $h'_{i+1,k} = h_{i+1,k}$ for $i = 0, \dots, n-1$, $k = 0, \dots, i$. We then have

$$\begin{aligned} h'_{n+1,k} &= \frac{\theta_{n+1}}{\Omega_{n+1}} \left(2\theta_k - \sum_{j=k+1}^n h'_{j,k} \right) = \frac{\theta_{n+1}}{\Omega_{n+1}} \left(2\theta_k - \sum_{j=k+1}^{n-1} h'_{j,k} - h'_{n,k} \right) \\ &= \frac{\theta_{n+1}}{\Omega_{n+1}} \left(\frac{\Omega_n}{\theta_n} h'_{n,k} - h'_{n,k} \right) = \frac{(\Omega_n - \theta_n) \theta_{n+1}}{\theta_n \Omega_{n+1}} h_{n,k} = h_{n+1,k}. \end{aligned}$$

Proposition 3 *The sequence $\{\mathbf{x}_0, \dots, \mathbf{x}_N\}$ generated by Algorithm FSFOM with (58) is identical to the corresponding sequence generated by Algorithm GOGM1.*

Proof We use induction, and for clarity, we use the notation $\mathbf{x}'_0, \dots, \mathbf{x}'_N$ for Algorithm FSFOM with (58). It is obvious that $\mathbf{x}'_0 = \mathbf{x}_0$, and we have

$$\begin{aligned}\mathbf{x}'_1 &= \mathbf{x}'_0 - \frac{1}{L} h_{1,0} \nabla f(\mathbf{x}'_0) = \mathbf{x}_0 - \frac{1}{L} \left(1 + \frac{(2\theta_0 - 1)\theta_1}{\Omega_1} \right) \nabla f(\mathbf{x}_0) \\ &= \mathbf{y}_1 + \frac{(\Omega_0 - \theta_0)\theta_1}{\theta_0 \Omega_1} (\mathbf{y}_1 - \mathbf{y}_0) + \frac{(2\theta_0^2 - \Omega_0)\theta_1}{\theta_0 \Omega_1} (\mathbf{y}_1 - \mathbf{x}_0) = \mathbf{x}_1.\end{aligned}$$

Assuming $\mathbf{x}'_i = \mathbf{x}_i$ for $i = 0, \dots, n$, we then have

$$\begin{aligned}\mathbf{x}'_{n+1} &= \mathbf{x}'_n - \frac{1}{L} h_{n+1,n} \nabla f(\mathbf{x}'_n) - \frac{1}{L} h_{n+1,n-1} \nabla f(\mathbf{x}'_{n-1}) - \frac{1}{L} \sum_{k=0}^{n-2} h_{n+1,k} \nabla f(\mathbf{x}'_k) \\ &= \mathbf{x}_n - \frac{1}{L} \left(1 + \frac{(2\theta_n - 1)\theta_{n+1}}{\Omega_{n+1}} \right) \nabla f(\mathbf{x}_n) - \frac{1}{L} \frac{(\Omega_n - \theta_n)\theta_{n+1}}{\theta_n \Omega_{n+1}} (h_{n,n-1} - 1) \nabla f(\mathbf{x}_{n-1}) \\ &\quad - \frac{1}{L} \sum_{k=0}^{n-2} \frac{(\Omega_n - \theta_n)\theta_{n+1}}{\theta_n \Omega_{n+1}} h_{n,k} \nabla f(\mathbf{x}_k) \\ &= \mathbf{y}_{n+1} - \frac{1}{L} \frac{(2\theta_n^2 - \Omega_n)\theta_{n+1}}{\theta_n \Omega_{n+1}} \nabla f(\mathbf{x}_n) \\ &\quad - \frac{1}{L} \frac{(\Omega_n - \theta_n)\theta_{n+1}}{\theta_n \Omega_{n+1}} \left(\nabla f(\mathbf{x}_n) - \nabla f(\mathbf{x}_{n-1}) + \sum_{k=0}^{n-1} h_{n,k} \nabla f(\mathbf{x}_k) \right) \\ &= \mathbf{y}_{n+1} + \frac{(2\theta_n^2 - \Omega_n)\theta_{n+1}}{\theta_n \Omega_{n+1}} (\mathbf{y}_{n+1} - \mathbf{x}_n) \\ &\quad + \frac{(\Omega_n - \theta_n)\theta_{n+1}}{\theta_n \Omega_{n+1}} \left(-\frac{1}{L} \nabla f(\mathbf{x}_n) + \frac{1}{L} \nabla f(\mathbf{x}_{n-1}) + \mathbf{x}_n - \mathbf{x}_{n-1} \right) \\ &= \mathbf{y}_{n+1} + \frac{(\Omega_n - \theta_n)\theta_{n+1}}{\theta_n \Omega_{n+1}} (\mathbf{y}_{n+1} - \mathbf{y}_n) + \frac{(2\theta_n^2 - \Omega_n)\theta_{n+1}}{\theta_n \Omega_{n+1}} (\mathbf{y}_{n+1} - \mathbf{x}_n) = \mathbf{x}_{n+1}.\end{aligned}$$

Proposition 4 The sequence $\{\mathbf{x}_0, \dots, \mathbf{x}_N\}$ generated by Algorithm FSFOM with (24) is identical to the corresponding sequence generated by Algorithm GOGM2.

Proof We use induction, and for clarity, we use the notation $\mathbf{x}'_0, \dots, \mathbf{x}'_N$ for Algorithm FSFOM with (24). It is obvious that $\mathbf{x}'_0 = \mathbf{x}_0$, and we have

$$\begin{aligned}\mathbf{x}'_1 &= \mathbf{x}'_0 - \frac{1}{L} h_{1,0} \nabla f(\mathbf{x}'_0) = \mathbf{x}_0 - \frac{1}{L} \left(1 + \frac{(2\theta_0 - 1)\theta_1}{\Omega_1} \right) \nabla f(\mathbf{x}_0) \\ &= \left(1 - \frac{\theta_1}{\Omega_1} \right) \left(\mathbf{x}_0 - \frac{1}{L} \nabla f(\mathbf{x}_0) \right) + \frac{\theta_1}{\Omega_1} \left(\mathbf{x}_0 - \frac{1}{L} \nabla f(\mathbf{x}_0) - \frac{1}{L} (2\theta_0 - 1) \nabla f(\mathbf{x}_0) \right) \\ &= \left(1 - \frac{\theta_1}{\Omega_1} \right) \mathbf{y}_1 + \frac{\theta_1}{\Omega_1} \mathbf{z}_1 = \mathbf{x}_1.\end{aligned}$$

Assuming $\mathbf{x}'_i = \mathbf{x}_i$ for $i = 0, \dots, n$, we then have

$$\begin{aligned}
\mathbf{x}'_{n+1} &= \mathbf{x}'_n - \frac{1}{L} h_{n+1,n} \nabla f(\mathbf{x}'_n) - \frac{1}{L} \sum_{k=0}^{n-1} h_{n+1,k} \nabla f(\mathbf{x}'_k) \\
&= \mathbf{x}_n - \frac{1}{L} \left(1 + \frac{(2\theta_n - 1)\theta_{n+1}}{\Omega_{n+1}} \right) \nabla f(\mathbf{x}_n) - \frac{1}{L} \sum_{k=0}^{n-1} \frac{\theta_{n+1}}{\Omega_{n+1}} \left(2\theta_k - \sum_{j=k+1}^n h_{j,k} \right) \nabla f(\mathbf{x}_k) \\
&= \left(1 - \frac{\theta_{n+1}}{\Omega_{n+1}} \right) \left(\mathbf{x}_n - \frac{1}{L} \nabla f(\mathbf{x}_n) \right) \\
&\quad + \frac{\theta_{n+1}}{\Omega_{n+1}} \left(\mathbf{x}_n - \frac{1}{L} \sum_{k=0}^n 2\theta_k \nabla f(\mathbf{x}_k) + \frac{1}{L} \sum_{k=0}^{n-1} \sum_{j=k+1}^n h_{j,k} \nabla f(\mathbf{x}_k) \right) \\
&= \left(1 - \frac{\theta_{n+1}}{\Omega_{n+1}} \right) \left(\mathbf{x}_n - \frac{1}{L} \nabla f(\mathbf{x}_n) \right) + \frac{\theta_{n+1}}{\Omega_{n+1}} \left(\mathbf{x}_0 - \frac{1}{L} \sum_{k=0}^n 2\theta_k \nabla f(\mathbf{x}_k) \right) \\
&= \left(1 - \frac{\theta_{n+1}}{\Omega_{n+1}} \right) \mathbf{y}_{n+1} + \frac{\theta_{n+1}}{\Omega_{n+1}} \mathbf{z}_{n+1}.
\end{aligned}$$

8.5 Proof of Lemma 4

It is obvious that $(\boldsymbol{\lambda}, \boldsymbol{\eta}, \boldsymbol{\tau}, \boldsymbol{\beta})$ in (40) and (41) is in \mathcal{A}'' (38). Using (37) and (57), we have

$$\begin{aligned}
&\mathbf{S}''(\mathbf{h}, \boldsymbol{\lambda}, \boldsymbol{\eta}, \boldsymbol{\tau}, \boldsymbol{\beta}) \\
&= \begin{cases} \frac{1}{2} \left((\lambda_i + \tau_i) h_{i,k} + \tau_i \sum_{j=k+1}^{i-1} h_{j,k} \right), & i = 2, \dots, N, \ k = 0, \dots, i-1, \\ \frac{1}{2} ((\lambda_i + \tau_i) h_{i,k} - \lambda_i), & i = 1, \dots, N, \ k = i-1, \\ \lambda_{i+1} - \beta_i, & i = 0, \dots, N-1, \ k = i, \\ \eta - \beta_N, & i = N, \ k = i, \end{cases}
\end{aligned} \tag{59}$$

and inserting (39), (40), and (41) yields

$$\begin{aligned}
&\mathbf{S}''(\mathbf{h}, \boldsymbol{\lambda}, \boldsymbol{\eta}, \boldsymbol{\tau}, \boldsymbol{\beta}) \\
&= \begin{cases} \frac{1}{2} \left(t_i^2 \tau_0 \frac{1}{t_i} \left(t_k - \sum_{j=k+1}^{i-1} h_{j,k} \right) + t_i \tau_0 \sum_{j=k+1}^{i-1} h_{j,k} \right), & i = 2, \dots, N, \ k = 0, \dots, i-1, \\ \frac{1}{2} \left(t_i^2 \tau_0 \left(1 + \frac{t_{i-1}-1}{t_i} \right) - t_{i-1}^2 \tau_0 \right), & i = 1, \dots, N, \ k = i-1, \\ \frac{1}{2} t_i^2 \tau_0, & i = 0, \dots, N, \ k = i, \end{cases} \\
&= \frac{1}{2} t_i t_k \tau_0, \quad i = 0, \dots, N, \ k = 0, \dots, i,
\end{aligned}$$

for t_i in (4). Then, using (40), we finally show that the feasibility condition of (D'') holds:

$$\begin{pmatrix} \mathbf{S}''(\mathbf{h}, \boldsymbol{\lambda}, \boldsymbol{\eta}, \boldsymbol{\tau}, \boldsymbol{\beta}) & \frac{1}{2} \boldsymbol{\tau} \\ \frac{1}{2} \boldsymbol{\tau}^\top & \frac{1}{2} \gamma \end{pmatrix} = \frac{1}{2} \mathbf{t} \mathbf{t}^\top \tau_0 \succeq 0,$$

where $\mathbf{t} = (t_0, \dots, t_N, 1)^\top$.

8.6 Proof of Lemma 5

It is obvious that $(\lambda, \eta, \tau, \beta)$ in (44) and (45) is in Λ'' . Inserting (27), (44) and (45) to (59) yields

$$\begin{aligned} \mathcal{S}''(\mathbf{h}, \lambda, \eta, \tau, \beta) &= \begin{cases} \frac{1}{2} \left(T_i \tau_0 \frac{t_i}{T_i} \left(2t_k - \sum_{j=k+1}^{i-1} h_{j,k} \right) + t_i \tau_0 \sum_{j=k+1}^{i-1} h_{j,k} \right), & i = 2, \dots, N, \quad k = 0, \dots, i-1, \\ \frac{1}{2} \left(T_i \tau_0 \left(1 + \frac{(2t_{i-1}-1)t_i}{T_i} \right) - T_{i-1} \tau_0 \right), & i = 1, \dots, N, \quad k = i-1, \\ T_i \tau_0 - (T_i - t_i^2) \tau_0, & i = 0, \dots, N, \quad k = i, \end{cases} \\ &= t_i t_k \tau_0, \quad i = 0, \dots, N, \quad k = 0, \dots, i, \end{aligned}$$

for t_i and T_i in (29). Then, using (44), we finally show that the feasibility condition of (D'') holds:

$$\begin{pmatrix} \mathcal{S}''(\mathbf{h}, \lambda, \eta, \tau, \beta) & \frac{1}{2} \tau \\ \frac{1}{2} \tau^\top & \frac{1}{2} \gamma \end{pmatrix} = \mathbf{t} \mathbf{t}^\top \tau_0 \succeq 0,$$

where $\mathbf{t} = (t_0, \dots, t_N, \frac{1}{2})^\top$.

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